Web Consequence Untangled

Stephan Krämer

Corresponding author(s). E-mail(s):
stephan.kraemer@uni-hamburg.de;

Abstract

Under the standard modal explication of consequence, a conclusion is a consequence of some premises just in case necessarily, if the latter are true, so is the former. Notoriously, this explication yields some results that at first glance are counter-intuitive. In particular, a necessary truth is a consequence of arbitrary premises, and premises that cannot all be true together entail arbitrary conclusions. In his paper 'On Ground and Consequence' (*Synthese*, 2021), Benjamin Schnieder introduces a novel notion of *web consequence*, defined on the basis of the concept of ground, which, he argues, fits our intuitive conception better. Building on his idea, the present paper examines the concept of web consequence in more detail. In particular, I provide three alternative, and simpler semantic characterizations of Schnieder's propositional logic of web consequence, two within a form of truthmaker semantics, one within a many-valued setting. I then consider some natural variations on that logic and establish their connections to well-known subclassical logics such as FDE, K_3, and LP. Finally, I provide sound and complete tableaux-based proof systems for each of the logics of web consequence so obtained.

Keywords: Logic, Relevance, Ground, Truthmaker Semantics, Many-Valued Logic

Preprint of a paper forthcoming in *Topoi*. Please cite from the original version at link.springer.com.

1 Introduction

The standard way of explicating the notion of consequence—of some conclusion following from some premises—is in modal terms: a conclusion is a consequence of some premises just in case necessarily, if the premises are true, so is the conclusion. As is well-known, this explication yields some prima facie peculiar instances of consequence. In particular, if it is impossible for some premises to be jointly true, then an arbitrary conclusion is a consequence of them, and if a conclusion is necessarily true, then it is a consequence of an arbitrary collection of premises. There is a rich literature on how to characterize formal theories of logical consequence that avoid such results, in which a conclusion is entailed[1] by some premises only if there is a relevant connection between premises and conclusion. But we are not usually given an alternative, general *explication* of consequence replacing the modal one, which might be seen as underlying those formal theories. In his recent paper 'On Ground and Consequence' (**?**), Benjamin Schnieder proposes such an alternative explication of consequence, which makes central use of the concept of *ground*. Roughly, the idea is to regard a conclusion as following from some premises just in case, however the premises might be grounded, they have a ground which contains a ground of the conclusion. Schnieder labels the conception of consequence so characterized *web consequence*, since consequence is taken to be a matter of the existence of suitable paths from premises to conclusion within the web of their potential grounds. He then proceeds to develop a formal implementation of this notion with respect to a propositional language, and to discuss some features of the resulting propositional logic, without however providing a sound and complete deductive system.

Building on Schnieder's work, my aim in this paper is to explore the concept of web consequence further. In the first part, I consider a natural variation, already hinted at by Schnieder, on the general definition of web consequence, and discuss its relation to Schnieder's original version. In the second part, I turn to the formal semantic characterization of web consequence. Here, I develop three alternative semantic characterizations of Schnieder's logic, all of which are simpler than Schnieder's and some of which are, in a sense to be explained, more robustly semantic. Two of them are set within the framework of truthmaker semantics, while the third one uses a many-valued semantics. I also consider some natural variations on Schnieder's logic and relate them to well-known subclassical logics such as FDE, K_3, and LP. In the third part, I turn to the proof theory, and provide sound and complete tableaux-based proof systems for each of the resulting logics of web consequence.

2 General Explications of Web Consequence

In this section, I introduce Schnieder's general definition of web consequence and explain how he motivates it (cf. (**?**, sec. 4)).[2] I go on to specify a variation on his definition which is hinted at by Schnieder and discuss how the definitions are related.

---

[1] I use the terms 'consequence', 'follows from', 'entails' and their ilk interchangeably, so that some premises entail a conclusion just in case the conclusion follows from, i.e. is a consequence of, the premises.

[2] I have reformulated some of the definitions in a way that I find clearer and easier to grasp. So my definitions below differ from Schnieder's in some superficial respects, but they are easily seen to be equivalent.

The counter-intuitive instances of modal consequence we have mentioned above are cases in which, intuitively, it seems that the premises are irrelevant to the conclusion. In light of this, it is a natural idea to try to use the notion of ground to obtain a more satisfactory explication of consequence.[3] After all, it is uncontroversial that in order for some facts to ground another, the putative grounds must be relevant to the putative groundee. Moreover, it is widely held that grounds must also *necessitate* what they ground, so that any instance of grounding is automatically an instance of modal consequence.[4]

So ground has some features we would like the desired explication of consequence to have. It also has some features that seem unsuitable for a consequence relation. Before stating them, let me introduce some symbolism to speak about ground. For present purposes, we may regard (full[5]) ground as a relation between a set of true propositions—the grounds—and a true proposition—the groundee. I shall use variables P, Q, ... to range over propositions and Γ, Δ, ... to range over sets of propositions, and accordingly write statements of ground like this: $\Gamma < P$.

The crucial 'unwelcome' features of ground are:

- *Non-Monotonicity*: $\Gamma < P$ does not entail $\Gamma \cup \Delta < P$
- *Factivity*: $\Gamma < P$ entails that P and all members of Γ are true
- *Directedness*: $\Gamma \cup \{Q\} < P$ entails $\Delta \cup \{P\} \not< Q$

Following Schnieder, I will assume for the purposes of this paper that none of these three features are acceptable for an explication of consequence.

Starting from the relation of ground, we can think of Schnieder's strategy as that of defining a notion of consequence by stripping ground of these three characteristics. To obtain a monotonic relation, Schnieder asks not if the premises ground of the conclusion, but if they *contain*, as a subset, a ground of the conclusion. To obtain a non-factive relation, he proposes to raise that question *under the hypothesis that the premises are true*. Finally, to obtain an undirected, i.e. non-hierarchical or *flat* relation, he has us consider not just the premises, but the premises *or any of their grounds*, as sources for a ground of the conclusion.

More precisely, we proceed as follows. First, we define relations of thin grounding both between a set of propositions and a proposition and between sets of propositions.

Definition 1 Let P, Q be propositions and Γ, Δ sets of propositions. Then

[3] I assume that readers are familiar with the basic features of ground; for a good introduction, see e.g. **?**, **?**.

[4] This assumption is widely but not universally accepted, dissenters include **??**. I shall assume throughout that grounds necessitate what they ground.

[5] There is a standard distinction between full and partial grounding. The simplest example to illustrate the distinction concerns conjunctive groundees. The fact that snow is white and the fact that grass is green *together* fully ground the fact that snow is white and grass is green. Each of the two facts that snow is white and that grass is green individually *partially* grounds the conjunctive fact, but is on its own insufficient to fully ground it.

- Γ thinly grounds (short: t-grounds) P iff Γ grounds P or ($\Gamma = \{P\}$ and P is true)
- Γ t-grounds Δ iff Γ is the union of t-grounds for each $P \in \Delta$

We may now define Schnieder's notion of web consequence—call it WCS for *w*eb *c*onsequence, *S*chnieder—as follows:

Definition 2 Let P be a proposition and Γ a set of propositions. Then Γ entails$_{\text{WCS}}$ P iff: in every scenario in which Γ is true, some t-ground Δ of Γ has a subset Δ^0 which t-grounds P.

Two comments on the phrase 'in every scenario in which Γ is true' are in order. First, for brevity, I call a set of propositions true iff every member of the set is true. Second, I have replaced Schnieder's phrase 'under the hypothesis that' by 'in every scenario in which', anticipating how Schnieder himself proposes to explicate the idea later on.[6] Note also that if we assume that t-grounds amalgamate, so that the union of some t-grounds is always itself a t-ground, the condition simplifies to: in every scenario in which Γ is true, the *maximal* t-ground of Γ contains a t-ground of P.

In addition to narrowly ground-theoretic resources, this explication of consequence invokes a kind of *suppositional* device in generalizing over alternative scenarios and asking us to evaluate what holds within these scenarios. The purpose of this is to 'de-factivize' ground: to allow premises to entail a conclusion even when premises and/or conclusion are false. Now some authors hold that there is also a non-factive understanding of the notion of ground itself. So a natural question is if we can avoid making use of the suppositional construction by appealing to a non-factive notion of ground instead.[7] The simplest suggestion is to delete the generalization over scenarios and replace t-ground by a non-factive version. So let us first define non-factive versions of t-ground.

Definition 3 Let P, Q be propositions and Γ, Δ sets of propositions. Then

- Γ thinly non-factively grounds (short: tn-grounds) P iff Γ non-factively grounds P or $\Gamma = \{P\}$
- Γ tn-grounds Δ iff Γ is the union of tn-grounds for each $P \in \Delta$

Can we now just take Γ to entail P iff some tn-ground of Γ contains a t-ground of P?

The answer is clearly negative. Given the standard assumption that a disjunction is non-factively grounded by each disjunct, this would mean that

[6] In Schnieder's formal account, the role of the phrase 'under the hypothesis that' is taken over by a universal generalization over models, and in his subsequent discussion of his logic, Schnieder glosses talk of models in terms of talk of scenarios; cf. his **?**, sec. 5.

[7] Indeed, Schnieder himself claims that this can be done, but prefers for reasons of familiarity to stick to factive ground and the suppositional construction; cf. his **?**, p. S1348.

$P \vee Q$ in general entails $P \wedge Q$, which it obviously does not. To see what has gone wrong, it is useful to consider more carefully what the suppositional construction in Schnieder's definition does. We may describe it as having a twofold effect. The first is one of *supply*: within the scope of 'in every scenario in which Γ is true', we have available the assumption that the premise set Γ is true. The second effect, however, is one of *subtraction*: within the scope of 'in every scenario in which Γ is true', we no longer have available any more specific information about how the premises are grounded, beyond the fact that we are in a scenario in which they are grounded somehow. Switching from a factive to a non-factive notion of ground is an alternative way to achieve the first effect, but not the second, equally crucial one.

This diagnosis also suggests a natural fix: instead of generalizing over scenarios in which some non-factive ground of the premises is also a factive ground, we may simply generalize directly over all non-factive grounds. That is, instead of generalizing over all scenarios in which the premise set Γ is true, i.e. which features some t-ground of Γ, we generalize directly over all tn-grounds of Γ. And instead of asking of each such scenario whether it features a t-ground of Γ that contains a t-ground of the conclusion P, we ask of any such tn-ground whether it has a tn-ground that contains a tn-ground of P. The resulting notion of web consequence—call it WCN for *w*eb *c*onsequence, *n*on-factive—is then as follows:

Definition 4 Let P be a proposition and Γ a set of propositions. Then Γ entails$_{\text{WCN}}$ P iff: every tn-ground Γ' of Γ has a tn-ground Δ which has a subset Δ^0 which tn-grounds P.

For a number of reasons, it is not straightforward to say how this explication of web consequence relates to the previous one; it depends on exactly how one conceives of scenarios as well as non-trivial details about the structural features of ground, non-factive ground, and their relation. As we shall see in the subsequent sections, however, once we have fixed the central *logical* features of scenarios, the relationships between the (propositional) *logics* of these consequence relations can be answered somewhat more straightforwardly.[8]

3 The Ground-Theoretic Semantics

In this section, we turn to the formal implementation of our two definitions of web consequence for the case of propositional logic. I will present a slightly more general version of Schnieder's formal semantics and use it to define four candidate implementations of WCS, one of which coincides with Schnieder's, as well as a formal implementation of WCN.

[8] Roughly speaking, the general question becomes as manageable as the logical one if we assume that grounding chains always terminate, and that whenever a proposition with non-factive grounds is true in a scenario, so is at least one of its non-factive grounds. The reason that the logical question is simpler is that these assumptions are unproblematic for the cases of logical grounding relevant to the logic of the consequence relations.

Let L be a propositional language with connectives $\wedge$, $\vee$, and $\neg$, based on a set of sentence letters (short: atoms) At. As usual, atoms and their negations are called literals, and their set is denoted by Lit. We use variables p, q, r for arbitrary atoms, ϕ, ψ, χ for arbitrary literals, A, B, C for arbitrary formulas, and Γ, Δ for arbitrary sets of formulas of L, adding various adornments like primes, sub-, or superscripts as we see fit.

We use standard notions of completeness, consistency, and classicality for sets of literals:

Definition 5 Let $s \subseteq Lit$. Then

- s is *complete* iff for every $p \in At$, $s \cap \{p, \neg p\}$ has at least one member
- s is *consistent* iff for every $p \in At$, $s \cap \{p, \neg p\}$ has at most one member
- s is *classical* iff s is complete and consistent

The central element of Schnieder's semantics is a recursive definition of a (thin) grounding relation over L relative, in effect, to a set s of literals. The definition may be stated as follows.[9]

Definition 6 For $s \subseteq Lit$, let $\leq_s$ be the smallest subset of $\wp(L) \times L$ satisfying the following conditions, where $G_s(A)$ abbreviates that there is some $\Gamma \subseteq L$ with $\Gamma \leq_s A$:

- $\{\phi\} \leq_s \phi$ if $\phi \in s$
- $\{A\} \leq_s A \vee B$ and $\{A\} \leq_s B \vee A$ if $G_s(A)$
- $\{A, B\} \leq_s A \vee B$ and $\{A, B\} \leq_s A \wedge B$ if $G_s(A)$ and $G_s(B)$
- $\{\neg A\} \leq_s \neg(A \wedge B)$ and $\{\neg A\} \leq_s \neg(B \wedge A)$ if $G_s(\neg A)$
- $\{\neg A, \neg B\} \leq_s \neg(A \wedge B)$ and $\{\neg A, \neg B\} \leq_s \neg(A \vee B)$ if $G_s(\neg A)$ and $G_s(\neg B)$
- $\{A\} \leq_s \neg\neg A$ if $G_s(A)$
- $\{A\} \leq_s A$ if $G_s(A)$
- $\Gamma \cup \Delta \leq_s A$ if $\Gamma \leq_s B$ and $\{B\} \cup \Delta \leq_s A$

I should note that Schnieder's definition (**?**, p. S1350) differs from the present one with respect to some details. First, Schnieder's definition is relative to what he counts as a model, which corresponds to a *complete* set of literals. But the completeness requirement plays no role in the definition, and our purposes are better served by a more general definition. Secondly, Schnieder simultaneously defines by induction three distinct items: a property G—being grounded—of formulas, a grounding relation corresponding to my $\leq$, and a partial version of it. It is not completely transparent to me how exactly his definition is to be understood, so I prefer to use the present one. Indeed, there seem to

[9] An extensionally similar, syntactic definition of a (logical) grounding relation over a propositional language is given in (**?**, sec. 1). Correia's definition is in terms of certain kinds of trees, which are constructed by rules for the different types of sentences that roughly mirror the clauses for the same types of sentences in the definition below. There are other interesting points of contact between Correia's work and the material below, but a more thorough comparison is unfortunately beyond the scope of this paper.

be at least minor problems with Schnieder's own definition. In particular, in place of my final clause, Schnieder has a simple transitivity clause for one-one partial ground. But this clause only ever yields new instances of partial ground, whereas Schnieder needs it to yield instances of full ground (cf. e.g. his proof that $(A \wedge B) \vee (A \wedge C)$ entails $A \wedge (B \vee C)$; (**?**, p. S1352)). The present definition affords a straightforward way to avoid these issues, and it seems clear that it accords with Schnieder's intentions.

If we think of s as the set of true literals under a given interpretation of L, $\leq_s$ is a relation of thin, full, factive grounding. So, taking sets of literals as models of scenarios, we might obtain a formal counterpart for L to Schnieder's WCS by taking Γ to entail C just in case for every $s \subseteq Lit$, if $G_s(\Gamma)$, then there are $\Delta^0 \subseteq \Delta \subseteq L$ such that $\Delta \leq_s \Gamma$ and $\Delta^0 \leq_s C$. Now note that grounds in the sense of $\leq_s$ amalgamate. For suppose that $\Gamma \leq_s A$ and $\Delta \leq_s A$. It follows that $G_s(A)$ and hence by the penultimate clause of the definition, $\{A\} \cup \{A\} \leq_s A$. Two applications of the final Cut clause then yield $\Gamma \cup \Delta \leq_s A$, as required. As a result, the condition that there are $\Delta^0 \subseteq \Delta \subseteq L$ such that $\Delta \leq_s \Gamma$ and $\Delta^0 \leq_s C$ simplifies to: the maximal $\Delta \subseteq L$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$.

To give precise and explicit definitions of consequence, we start with two auxiliary definitions. Firstly, we state a precise definition of the grounding relation, relativized to a set of literals s, between sets of formulas and sets of formulas in terms of the basic relation between sets of formulas and formulas. Secondly, we define a relation, also relative to a set of literals s, that obtains just in case the simplified condition just described is satisfied, i.e. just in case there exists a suitable path along the s-relative grounding web from the premises Γ to conclusion C:

Definition 7 Let $s \subseteq Lit$, $C \in L$, $\Gamma \subseteq L$. Then

- $\Delta \leq_s \Gamma$ iff: for some indexing $\{A_i\}_{i \in I}$ of Γ, there is an indexed family of sets $\{\Delta_i\}_{i \in I}$ with $\Delta = \bigcup_{i \in I} \Delta_i$ and $\Delta_i \leq_s A_i$ for all $i \in I$
- $\Gamma \Rightarrow_s C$ iff: if $G_s(\Gamma)$, then the maximal $\Delta \subseteq L$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$.

In terms of this relation, we can then define different consequence relations by generalizing over different classes of sets of literals. I shall consider four such classes, comprising all classical, all complete, all consistent, and absolutely all sets of literals, respectively:

Definition 8 Let $\Gamma \cup \{C\} \subseteq L$. Then

- $\Gamma \models_4 C$ iff: $\Gamma \Rightarrow_s C$ for every $s \subseteq Lit$
- $\Gamma \models_I C$ iff: $\Gamma \Rightarrow_s C$ for every complete $s \subseteq Lit$
- $\Gamma \models_P C$ iff: $\Gamma \Rightarrow_s C$ for every consistent $s \subseteq Lit$
- $\Gamma \models_2 C$ iff: $\Gamma \Rightarrow_s C$ for every classical $s \subseteq Lit$

(In what is hopefully a helpful mnemonic, the subscripts are supposed to remind one of *4*-valued, possibly *i*nconsistent, possibly *p*artial, and *2*-valued interpretations.)

Relating these definitions to WCS, we may note that the work of the suppositional operator 'under the hypothesis that Γ is true' or 'in every scenario in which Γ is true' is now taken over by the device of generalizing over sets of literals, together with the conditionalization on the truth of Γ in the definition of $\Rightarrow_s$. Conditionalizing on the truth of Γ serves to make available the assumption that Γ is true. Generalizing over a class of sets of literals serves to make unavailable any assumptions about which statements are true which is not stable across all sets of literals in the class. So generalizing over all sets of literals, as in the definition of $\models_4$, removes *all* information about which statements are true. Generalizing only over the classical sets of literals, by contrast, retains every classical logical truth. Generalizing only over consistent sets retains the information that we never have both an atom and its negation true, and generalizing only over complete sets retains the information that we always have at least one of any atom and its negation true.

This last option is Schnieder's choice: his models are required to make true at least one of each atom and its negation, and his definition of consequence is readily seen to be equivalent to $\models_I$. But it is not immediately clear why this should be our choice, and not one of the other three. From both a technical and a philosophical point of view, it seems worthwhile also to consider these alternative definitions.

Let us finally see how the alternative general notion of web consequence WCN may be given a formal implementation. First, we shall need a non-factive version of the grounding relation defined above. The obvious modification of the definition is simply to remove all occurrences of 'if $G_s(A)$' and 'if $G_s(B)$', and to let $\{\phi\} \leq_s \phi$ hold irrespective of whether $\phi \in s$. The relation so defined is then independent of the choice of s, as one would expect, so we shall simply write $\leq_N$ for the resulting relation. We can then define non-factive versions of the set-set relation of ground and of $\Rightarrow_s$, and in terms of these, our consequence relation:

Definition 9 Let $s \subseteq Lit$ and $\Gamma \cup \{C\} \subseteq L$. Then

- $\Gamma \Rightarrow_N C$ iff the maximal $\Delta \subseteq L$ with $\Delta \leq_N \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_N C$
- $\Gamma \models_N C$ iff: for every $\Gamma' \leq_N \Gamma$, $\Gamma' \Rightarrow_N C$

We briefly pause to note a distinctive implication of all five of the above definitions of consequence: nothing is ever entailed by the empty set of premises.[10] To see this, note first that by definition 7, since the only indexing of the empty set is empty, and the empty union is the empty set, for all $s \subseteq Lit$, $\emptyset$ is the maximal $\Delta \subseteq L$ with $\Delta \leq_s \emptyset$. Moreover, for all $s \subseteq Lit$ and all $C \in L$, $\emptyset \not\leq_s C$.

[10] Schnieder also recognizes this feature of his account; cf. **?**, p. S1354.

As a result, $\emptyset \Rightarrow_s C$ fails for all $s \subseteq Lit$ and $C \in L$. It follows that $\emptyset \not\models_* C$ for all $* \in \{4, I, P, 2\}$; parallel reasoning applies in the case of $\models_N$.

As Schnieder's own discussion suggests (cf. **?**, pp. S1352ff), it is possible to simplify his semantics somewhat. Note first that under the present definition of $\leq_s$, $\Gamma \leq_s C$ implies that Γ is finite, and every formula has only finitely many distinct grounds. Moreover, as Schnieder emphasizes, it is straightforward to verify that under this definition, whenever some formula is grounded at all, it is grounded by a set of literals. This is immediate for literals, and given the final Cut clause in the definition, it is straightforward to show that the property of being grounded by a set of literals if grounded at all is preserved under conjunction, disjunction, negated conjunction, negated disjunction, as well as double negation. As a result, we can in general restrict attention to grounding relationships with sets of literals as grounds. This allows us to simplify the definitional conditions for $\Rightarrow_s$ and $\Rightarrow_N$.

Proposition 1 *Let $\leq_s^l$ and $\leq_N^l$ be the restriction of $\leq_s$ and $\leq_N$ to $\wp(Lit) \times L$. Let $\Gamma \cup \{C\} \subseteq L$ and $s \subseteq Lit$. Then*

1. *$\Gamma \Rightarrow_s C$ iff: if $G_s(\Gamma)$ then the maximal $\Delta \subseteq Lit$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$*
2. *$\Gamma \Rightarrow_N C$ iff: the maximal $\Delta \subseteq Lit$ with $\Delta \leq_N \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_N C$*

Proof For (1), right-to-left, assume that $G_s(\Gamma)$, so by the right-hand side, the maximal $\Delta^l \subseteq Lit$ with $\Delta^l \leq_s \Gamma$ has a subset Δ^{l0} with $\Delta^{l0} \leq_s C$. Let Δ be the maximal subset of L with $\Delta \leq_s \Gamma$. Then $\Delta^l \subseteq \Delta$ and hence $\Delta^{l0} \subseteq \Delta$, so Δ has a subset Δ^0 with $\Delta^0 \leq_s C$, as required.

For the left-to-right direction, suppose $\Gamma \Rightarrow_s C$. Suppose $G_s(\Gamma)$, and let Δ be the maximal subset of L with $\Delta \leq_s \Gamma$. Then by the left-hand side, Δ has a subset Δ^0 with $\Delta^0 \leq_s C$. Now let Δ^l be the maximal subset of Lit with $\Delta^l \leq_s \Gamma$. Now pick a $\Delta^{l0} \subseteq Lit$ with $\Delta^{l0} \leq_s \Delta^0$. Since Δ^0 is finite, it follows by Cut that $\Delta^{l0} \leq_s C$. Now note that Δ^{l0} is the union of a finite family of grounds of elements of Δ. By Cut and maximality of Δ, each ground of an element of Δ is a subset of Δ, so Δ^{l0} is a subset of Δ, and hence a subset of Δ^l. So Δ^l has a subset that grounds C, as required.

For (2), right-to-left, suppose that the maximal $\Delta^l \subseteq Lit$ with $\Delta^l \leq_N \Gamma$ has a subset Δ^{l0} with $\Delta^{l0} \leq_N C$. Let Δ be the maximal subset of L with $\Delta \leq_N \Gamma$. Then $\Delta^l \subseteq \Delta$ and hence $\Delta^{l0} \subseteq \Delta$, so Δ has a subset Δ^0 with $\Delta^0 \leq_N C$, as required.

For the left-to-right direction, suppose $\Gamma \Rightarrow_N C$. Let Δ be the maximal subset of L with $\Delta \leq_N \Gamma$, so by the left-hand-side, Δ has a subset Δ^0 with $\Delta^0 \leq_N C$. Now let Δ^l be the maximal subset of Lit with $\Delta^l \leq_N \Gamma$. Pick a $\Delta^{l0} \subseteq Lit$ with $\Delta^{l0} \leq_s \Delta^0$. By the same reasoning as before, Δ^{l0} is then a subset of Δ^l with $\Delta^{l0} \leq_s C$, as required. $\square$

In addition, as is tedious but straightforward to verify, the restrictions $\leq_s^l$ and $\leq_N^l$ of $\leq_s$ and $\leq_N$ to $\wp(Lit) \times L$ can also be characterized as follows:

Proposition 2 *Let $\Gamma \cup \{s\} \subseteq Lit$. Then $\Gamma \leq^l_s A$ iff $\Gamma \subseteq s$ and $\Gamma \leq^l_N A$. And*

1. $\Gamma \leq^l_N \phi$ *iff* $\Gamma = \{\phi\}$
2. $\Gamma \leq^l_N A \vee B$ *iff* $\Gamma \leq^l_N A$ *or* $\Gamma \leq^l_N B$ *or* $\Gamma = \Gamma_A \cup \Gamma_B$ *for some* $\Gamma_A \leq^l_N A$ *and* $\Gamma_B \leq^l_N B$
3. $\Gamma \leq^l_N A \wedge B$ *iff* $\Gamma = \Gamma_A \cup \Gamma_B$ *for some* $\Gamma_A \leq^l_N A$ *and* $\Gamma_B \leq^l_N B$
4. $\Gamma \leq^l_N \neg(A \wedge B)$ *iff* $\Gamma \leq^l_N \neg A$ *or* $\Gamma \leq^l_N \neg B$ *or* $\Gamma = \Gamma_A \cup \Gamma_B$ *for some* $\Gamma_A \leq^l_N \neg A$ *and* $\Gamma_B \leq^l_N \neg B$
5. $\Gamma \leq^l_N \neg(A \vee B)$ *iff* $\Gamma = \Gamma_A \cup \Gamma_B$ *for some* $\Gamma_A \leq^l_N \neg A$ *and* $\Gamma_B \leq^l_N \neg B$
6. $\Gamma \leq^l_N \neg\neg A$ *iff* $\Gamma \leq^l_N A$

Proof Omitted for reasons of space. □

This fact can be used to show that the sets of literals that are non-factive grounds of a formula A of L are precisely the same as the *exact truthmakers* of A within the so-called canonical model of L within truthmaker semantics. This suggests that our five consequence relations can conveniently be studied within the general framework of truthmaker semantics. The next section pursues this idea.

4 Truthmaker Semantics

In this section, after briefly introducing the framework of truthmaker semantics, I define its canonical model for L and use it to obtain simplified characterizations of our five consequence relations (sec. 4.1). I go on to examine their relations to one another and to some other well-known consequence relations (sec. 4.2). Finally, I address the concern that both the definitions given within Schnieder's setting and those given in terms of the canonical model seem disappointingly linguistic by providing equivalent definitions in terms of a class of (non-canonical) models (sec. 4.3).

Much as in possible world semantics, a proposition is characterized in terms of the possible worlds at which it is true, in truthmaker semantics, a proposition is characterized in terms of the *states* that *make* it true. Like worlds, states are taken to be non-disjunctive in the sense that a state makes true a disjunction only if it makes true at least one of the disjuncts. Unlike worlds, states are allowed to be *partial*, leaving open the truth-value of some propositions. And unlike worlds, states are allowed to be *impossible*, so for a given proposition, a state may contain both a truthmaker of that proposition and a truthmaker of its negation.

Truthmaking, or *verification*, is taken to be *exact* in the sense that a truthmaker has to be *wholly relevant* to any proposition it makes true. For example, the state of it being windy and cold does not make true the proposition that it is windy, since it is partially irrelevant: it contains the state of it being cold, which is irrelevant to the truth of the proposition that it is windy. So, like ground, truthmaking is non-monotonic: a state containing a truthmaker of a proposition is not in general itself a truthmaker of that proposition, just as a collection of facts containing as a subcollection a ground of a fact is not in general itself a ground of that fact. More generally, the notion of truthmaking may be regarded as a kind of semantic reflection of the notion of ground: roughly speaking, for a state to make a proposition true is for the state's obtaining to non-factively ground that the proposition is true (cf. e.g. (**?**, sec. 3)). Correspondingly, as we shall see, the way truthmaker semantics takes verification to interact with conjunction, disjunction, and negation is very similar to the way Schnieder's semantics takes grounding to interact with the same operations.

The basic formal structure within truthmaker semantics is that of a *state-space*, defined as follows.[11]

Definition 10 A *state-space* is a pair $(S, \sqsubseteq)$ where S is a non-empty set and $\sqsubseteq$ is a partial order on S such that every subset T of S has a least upper bound wrt $\sqsubseteq$, denoted by $\bigsqcup T$.

Informally, we think of S as the set of states, and $\sqsubseteq$ as the relation of part-whole on the states. Where $T \subseteq S$, $\bigsqcup T$ is the fusion of the elements of T, and when $T = \{t_1, t_2, \ldots\}$ we also write $\bigsqcup T$ as $t_1 \sqcup t_2 \sqcup \ldots$ Any state-space has a least element, $\bigsqcup \emptyset$, which is called the nullstate and denoted by $\square$, as well as a greatest element, $\bigsqcup S$, which is called the fullstate and denoted by $\blacksquare$.

Definition 11 Let $(S, \sqsubseteq)$ be a state-space. Then

- A *unilateral* proposition is any non-empty subset of S
- For P, Q unilateral proposition, let
  - $P \wedge Q = \{s \sqcup t : s \in P, t \in Q\}$
  - $P \vee Q = P \cup Q \cup (P \wedge Q)$
- A *bilateral* proposition P is any pair of unilateral propositions; we refer to the first (second) coordinate of P as P^+ (P^-)
- For P, Q bilateral propositions, let
  - $\neg P = (P^-, P^+)$
  - $P \wedge Q = (P^+ \wedge Q^+, P^- \vee Q^-)$
  - $P \vee Q = (P^+ \vee Q^+, P^- \wedge Q^-)$

Note that from these definitions we can derive principles for (exact) verification by a state precisely analogous to the principles given in proposition 2 for being grounded by a set of literals:

Proposition 3 *Let $(S, \sqsubseteq)$ be a state-space, $s \in S$, and P and Q bilateral propositions from $(S, \sqsubseteq)$.*

[11] For a more detailed exposition of the framework, see **???**.

1. *s verifies P iff $s \in P^+$*
2. *s verifies $P \vee Q$ iff s verifies P or s verifies Q or $s = t \sqcup u$ for some $t \in S$ verifying P and $u \in S$ verifying Q*
3. *s verifies $P \wedge Q$ iff $s = t \sqcup u$ for some $t \in S$ verifying P and $u \in S$ verifying Q*
4. *s verifies $\neg(P \wedge Q)$ iff s verifies $\neg P$ or s verifies $\neg Q$ or $s = t \sqcup u$ for some $t \in S$ verifying $\neg P$ and $u \in S$ verifying $\neg Q$*
5. *s verifies $\neg(P \vee Q)$ iff $s = t \sqcup u$ for some $t \in S$ verifying $\neg P$ and $u \in S$ verifying $\neg Q$*
6. *s verifies $\neg\neg P$ iff s verifies P*

Proof Straightforward by application of the definitions. □

Based on any given state-space, we can now build *state-models* for our language L.

Definition 12 A state-model M for L is any triple $(S, \sqsubseteq, [\cdot])$ where $(S, \sqsubseteq)$ is a state-space and $[\cdot]$ is a function mapping each atom $p \in At$ to a bilateral proposition on $(S, \sqsubseteq)$.

The function $[\cdot]$ is extended to complex formulas in the obvious way, letting $[\neg A] = \neg[A]$, $[A \wedge B] = [A] \wedge [B]$, and $[A \vee B] = [A] \vee [B]$.

Given a state-model $M = (S, \sqsubseteq, [\cdot])$, we say that a state $s \in S$ exactly verifies$_M$ (falsifies$_M$) a formula $A \in L$ iff $s \in [A]^+$ ($[A]^-$), dropping the subscripts when no confusion threatens. We further say that s inexactly verifies (falsifies) A iff s has a part that exactly verifies (falsifies) A. When speaking of verification or falsification without qualification, I always mean the exact variant.

Finally, we extend the notions of verification to sets of formulas. When $\Gamma \subseteq L$, we call a *verifier-choice from* Γ any set of states that may be obtained by picking one verifier for each $A \in \Gamma$. A state s is then said to verify Γ iff s is the fusion of some verifier-choice from Γ. Note that if Γ is empty, the empty set is the sole verifier-choice, and so $\square$ is the sole verifier of Γ. A state is said to inexactly verify Γ iff it contains a verifier of Γ.

4.1 Canonical Semantics

For a given propositional language, there is a natural way to construct a state-space from the set of literals of the language; we call that state-space the canonical state-space for the language in question.

Definition 13

- The *canonical state-space* C for L is the pair $(S^C, \sqsubseteq^C)$ where $S^C = \wp(Lit)$ and $\sqsubseteq^C$ is the subset relation on S^C.

- The *canonical model* M^C for L is $(S^C, \sqsubseteq^C, [\cdot]^C)$ where $[\cdot]^C$ maps any atom p into the ordered pair $(\{\{p\}\}, \{\{\neg p\}\})$.

Relative to the canonical model, we can now give analogous definitions of consequence relations which are provably equivalent to the previous definitions.

Definition 14 Let $s \in S^C$ and $\Gamma \cup \{C\} \subseteq L$. Then $\Gamma \Rightarrow_s^C C$ iff: if s inexactly verifies Γ, then the maximal part of s verifying Γ contains a verifier of C.

Definition 15 Let $\Gamma \cup \{C\} \subseteq L$. Then

- $\Gamma \models_{CN} C$ iff in M^C, every verifier of Γ contains a verifier of C
- $\Gamma \models_{C4} C$ iff in M^C, $\Gamma \Rightarrow_s C$ for every $s \in S^C$
- $\Gamma \models_{CI} C$ iff in M^C, $\Gamma \Rightarrow_s C$ for every complete $s \in S^C$
- $\Gamma \models_{CP} C$ iff in M^C, $\Gamma \Rightarrow_s C$ for every consistent $s \in S^C$
- $\Gamma \models_{C2} C$ iff in M^C, $\Gamma \Rightarrow_s C$ for every classical $s \in S^C$

(In what follows, I shall often leave the qualification 'in M^C' implicit.)

Proposition 4 *For any $s \subseteq Lit$, $A \in L$, and $\Gamma \subseteq L$:*

1. s verifies A iff $s \leq_N^l A$,
2. s falsifies A iff $s \leq_N^l \neg A$,
3. s verifies Γ iff $s \leq_N^l \Gamma$,
4. $G_s(A)$ iff s inexactly verifies A,
5. $G_s(\Gamma)$ iff s inexactly verifies Γ

Proof By a straightforward induction on the complexity of A. □

Proposition 5 *Let $\Gamma \cup \{C\} \subseteq L$. Then $\Gamma \models_N C$ iff $\Gamma \models_{CN} C$*

Proof From propositions 2 and 4. □

Lemma 6 *Let $s \in S^C$ and $\Gamma \cup \{C\} \subseteq L$. Then $\Gamma \Rightarrow_s C$ iff $\Gamma \Rightarrow_s^C C$*

Proof From proposition 2(1), it follows that (*) for $\Delta \subseteq Lit$, $G_s(\Delta)$ iff $\Delta \subseteq s$.

Right-to-left: Suppose that $\Gamma \Rightarrow_s^C C$, so if s inexactly verifies Γ, then the maximal part of s verifying Γ contains a verifier of C. We need to show that $\Gamma \Rightarrow_s C$, i.e. that if $G_s(\Gamma)$, then the maximal $\Delta \subseteq Lit$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$. So suppose $G_s(\Gamma)$. By proposition 4, s inexactly verifies Γ. So by assumption, the maximal part t of s verifying Γ contains a verifier of C. By proposition 4, $t \leq_N \Gamma$, and for some $t' \sqsubseteq t$, $t' \leq_N C$. Since $t' \subseteq t \subseteq s$, it follows that $t \leq_s \Gamma$ and $t' \leq_s C$. From this, it follows straightforwardly that $\Gamma \Rightarrow_s C$, as required.

Now suppose that $\Gamma \Rightarrow_s C$, so if $G_s(\Gamma)$, then the maximal $\Delta \subseteq Lit$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$. We need to show that if s inexactly verifies Γ, then the maximal part of s verifying Γ contains a verifier of C. So suppose s inexactly verifies Γ. Then by proposition 4, $G_s(\Gamma)$. So by assumption, the maximal $\Delta \subseteq Lit$ with $\Delta \leq_s \Gamma$ has a subset Δ^0 with $\Delta^0 \leq_s C$. Since $\Delta \leq_s \Gamma$, we have $\Delta \subseteq s$ and $\Delta \leq_N \Gamma$, so by proposition 4, Δ verifies Γ, and hence is part of the maximal part of s verifying Γ. Since $\Delta^0 \leq_s C$, Δ^0 verifies C, so Δ, and hence the maximal part of s verifying Γ, contains a verifier of C, as required. □

Proposition 7 *Let* $\Gamma \cup \{C\} \subseteq L$. *Then* $\Gamma \models_* C$ *iff* $\Gamma \models_{C*} C$ *for all* $* \in \{4, I, P, 2\}$

Proof From lemma 6. □

4.2 Connections

In this section, we make a number of observations regarding the connections between the various consequence relations we have defined, as well as connections to other well-known consequence relations. First of all, it is straightforward to show that $\models_4$ and $\models_N$ coincide.

Proposition 8 $\Gamma \models_N C$ *iff* $\Gamma \models_4 C$

Proof Suppose that $\Gamma \models_4 C$, so for every $s \in S^C$, $\Gamma \Rightarrow_s C$. Let t be an arbitrary verifier of Γ. Then the maximal part of t verifying Γ contains a verifier of C. So t contains a verifier of C. By proposition 5, $\Gamma \models_N C$. Conversely, suppose $\Gamma \models_N C$, so every verifier of Γ contains a verifier of C. Let s be any inexact verifier of Γ, and let t be an exact verifier of Γ contained in s. Then t is contained in the maximal part of s verifying Γ. By assumption, t contains a verifier of C, and hence the maximal part of s verifying Γ contains a verifier of C. So $\Gamma \models_4 C$. □

Moreover, in terms of canonical models, and using our notions of arbitrary, complete, consistent, and classical canonical states, we can characterize four well-known consequence relations: first-degree entailment (FDE), the logic of paradox (LP), strong Kleene logic (K_3), and classical logic (CL):[12]

Definition 16 Let $\Gamma \cup \{C\} \subseteq L$. Then

[12]The most straightforward way to show this is to use the standard relational or four-valued semantics for these consequence relations (e.g. (**?**, ch. 7)), and to note the correspondence between an (arbitrary, complete, consistent, classical) state containing a verifier of a formula and (arbitrary, complete, consistent, classical) relational interpretations relating the formula to the truth-value 1.—(**?**, sec. 1-3) establishes related results, characterizing the same consequence relations in terms of a syntactically defined grounding relation that agrees in relevant respects with our verification relation in the canonical model.

- $\Gamma \models_{FDE} C$ iff for every $s \in S^C$, if s contains a verifier of Γ, s contains a verifier of C
- $\Gamma \models_{LP} C$ iff for every complete $s \in S^C$, if s contains a verifier of Γ, s contains a verifier of C
- $\Gamma \models_{K3} C$ iff for every consistent $s \in S^C$, if s contains a verifier of Γ, s contains a verifier of C
- $\Gamma \models_{CL} C$ iff for every classical $s \in S^C$, if s contains a verifier of Γ, s contains a verifier of C

$\models_{FDE}$ and $\models_{K3}$ coincide with $\models_4$ and $\models_P$, respectively:

Proposition 9 *Let* $\Gamma \cup \{C\} \subseteq L$. *Then*

1. $\Gamma \models_4 C$ *iff* $\Gamma \models_{FDE} C$
2. $\Gamma \models_P C$ *iff* $\Gamma \models_{K3} C$

Proof For 1., note that the condition that every verifier of Γ contains a verifier of C is equivalent to the condition that every state containing a verifier of Γ contains a verifier of C. The result then follows from proposition 8.

For 2., suppose $\Gamma \models_P C$, so $\Gamma \Rightarrow_s C$ for every consistent $s \in S^C$. Let $t \in S^C$ be consistent and suppose t contains a verifier of Γ. Then by assumption, the maximal part of t verifying Γ contains a verifier of C. So t contains a verifier of C, and so $\Gamma \models_{K3} C$. Suppose now that $\Gamma \models_{K3} C$, so every consistent state containing a verifier of Γ contains a verifier of C. Let s be a consistent state containing a verifier of Γ, and let t be the maximal verifier of Γ contained in s. Then t is a consistent state containing a verifier of Γ, and since $\Gamma \models_{K3} C$, t contains a verifier of C. So $\Gamma \models_P C$. □

$\models_{LP}$ and $\models_{CL}$, on the other hand, do not coincide with their web-consequence counterparts $\models_I$ and $\models_2$, but they are nevertheless closely related to them in the following ways:

Proposition 10 *Let* $\Gamma \cup \{C\} \subseteq L$. *Then*

1. $\Gamma \models_I C$ *implies* $\Gamma \models_{LP} C$
2. $\Gamma \models_2 C$ *implies* $\Gamma \models_{CL} C$
3. $\Gamma \models_{LP} C$ *does not imply* $\Gamma \models_I C$
4. $\Gamma \models_{CL} C$ *does not imply* $\Gamma \models_2 C$
5. $\Gamma \models_{LP} C$ *iff* $\Gamma, C \vee \neg C \models_I C$
6. $\Gamma \models_{CL} C$ *iff* $\Gamma, C \vee \neg C \models_2 C$

Proof For 1. and 2., suppose $\Gamma \models_I C$ / $\Gamma \models_2 C$, and let s be a complete / classical state containing a verifier of Γ. By assumption, the maximal part of s verifying Γ contains a verifier of C, and hence s contains a verifier of C, as required.

For 3. and 4., note that since any complete state contains a verifier of any instance of the law of excluded middle $C \vee \neg C$, we have $A \models_{LP} C \vee \neg C$ and $A \models_{CL} C \vee \neg C$ for arbitrary A and C. But since the maximal part of a complete state verifying A need not contain either a verifier of C or of $\neg C$, we have neither $A \models_I C \vee \neg C$ nor $A \models_2 C \vee \neg C$. In addition, as we already noted, we never have $\emptyset \models_I C$ or $\emptyset \models_2 C$, whereas $\emptyset \models_{LP} C$ and $\emptyset \models_{CL} C$ holds whenever C is a classical tautology.

For 5. and 6, suppose first that $\Gamma \models_{LP} C$ / $\Gamma \models_{CL} C$, and let s be a complete / classical state containing a verifier of $\Gamma, C \vee \neg C$. Then s contains a verifier of Γ and so by assumption, s contains a verifier of C. Then the maximal part of s containing a verifier of $\Gamma, C \vee \neg C$ also contains that verifier of C, so $\Gamma, C \vee \neg C \models_I C$ / $\Gamma, C \vee \neg C \models_2 C$.

Suppose now that $\Gamma, C \vee \neg C \models_I C$ / $\Gamma, C \vee \neg C \models_2 C$, and let s be a complete / classical state containing a verifier of Γ. Since s is complete, s also contains a verifier of $C \vee \neg C$, and hence of $\Gamma, C \vee \neg C$. By assumption, the maximal part of s containing a verifier of $\Gamma, C \vee \neg C$ contains a verifier of C, so s contains a verifier of C, and hence $\Gamma \models_{LP} C$ / $\Gamma \models_{CL} C$. □

Especially parts 5. and 6. of proposition 10 are striking. While $\models_I$ and $\models_2$ are both strictly speaking narrower relations of consequence than $\models_{LP}$ and $\models_{CL}$, whenever some conclusion follows from some premises according to the latter, it follows according to the former from the same premises, enriched by the instance of the law of excluded middle for the case of the conclusion. And note that even from the perspective of our various semantic characterizations of $\models_I$ and $\models_2$, there is a sense in which all instances of excluded middle are trivial: although they do not follow from the empty set of premises, they are verified—or grounded, in Schnieder's setting—within every model. For this reason, Schnieder even goes so far as to suggest that under the most appropriate understanding of the notion of a logical truth within this context, they should be counted as logical truths (cf. **?**, p. S1354). Informally, we might therefore perhaps describe the difference between $\models_I$ and $\models_{LP}$, and $\models_2$ and $\models_{CL}$ as follows. In order for $\Gamma \models_I C$ to hold, it needs to be the case that $\Gamma \models_{LP} C$ holds, and in addition, the premises Γ need to be in part *about* the subject matter of the conclusion C, although they do not need to provide non-trivial information about that subject matter—and likewise for $\models_2$ and $\models_{CL}$.

This is not to say, though, that the differences between these pairs of logics are merely superficial. One important difference is that for both LP and CL we have the so-called deduction theorem: if $\Gamma, A \models C$ then $\Gamma \models A \to C$. (Officially, our language does not include a material conditional, but we may take $A \to C$ to abbreviate $\neg A \vee C$.) For the web consequence relations, it is clear that this fails because never $\emptyset \models C$. In fact, we do not even need to consider the empty set of premises to show that the deduction theorem fails: it is readily verified that for all four web consequence relations, we have $A, B \models A \wedge B$ but $A \not\models B \to (A \wedge B)$, i.e. $A \not\models \neg B \vee (A \wedge B)$.

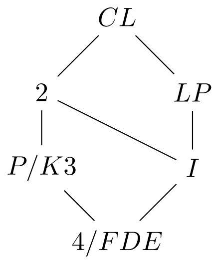
How do the consequence relations we have defined relate to each other, and how do they fare with respect to the counter-intuitive principles of classical logic mentioned at the beginning—i.e. the principle EFQ that a classical contradiction entails everything, and the principle VEQ that a classical tautology such as $A \vee \neg A$ is entailed by everything? To bring out the distinctive differences between the consequence relations, we may consider the following three principles, also including a weakened version of VEQ, on which $A \vee \neg A$ is entailed by the disjunction of itself with an arbitrary formula. For explicitness:

EFQ	$A \wedge \neg A$	$\models B$
VEQ	B	$\models A \vee \neg A$
WVEQ	$B \vee (A \vee \neg A)$	$\models A \vee \neg A$

The following table shows which principles hold for which consequence relation, where ✓ indicates that the principle in question does hold, and ✗ indicates that it does not.

	$\models_{4/FDE}$	$\models_I$	$\models_{LP}$	$\models_{P/K3}$	$\models_2$	$\models_{CL}$
EFQ	✗	✗	✗	✓	✓	✓
VEQ	✗	✗	✓	✗	✗	✓
WVEQ	✗	✓	✓	✗	✓	✓

So only $\models_4$ and $\models_I$ invalidate both EFQ and VEQ, and only $\models_4$ invalidates all three principles. The table also makes clear that none of the six consequence relations coincide. Putting our results so far together, we obtain the following view of their relation:



Here the lines represent relationships of strict inclusion, so that the consequence relation at the upper end of the line is strictly wider than the consequence relation at the lower end of the line.

Given that Schnieder's stated aim of invalidating EFQ and VEQ is achieved by both $\models_4$ and his preferred $\models_I$, it is natural to ask which of the two constitutes a more attractive account of consequence overall. Developing a well-supported answer to this question is beyond the scope of this paper. However, in the concluding section 7 below, I shall briefly return to this matter and highlight some points of contrast between the two accounts that seem particularly significant to me.

4.3 R-Space Semantics

There is a way in which both the semantics proposed by Schnieder and the canonical model semantics can seem objectionably non-semantic, as it were. In Schnieder's setting, the central ingredient in the definition of consequence is a syntactically defined relation of grounding. In the setting of truthmaker semantics, we switched to a more semantic sounding relation of verification, but so far we have appealed only to the canonical model, within which verification is still just a relation between formulas and sets of formulas and ultimately characterized in purely syntactic fashion.

One may debate to what extent this feature of the definitions should be considered problematic, but it seems worth asking whether we can also give an equivalent semantics which simply avoids this objection. And indeed we can, in a number of ways. In this section, I present a more general form of truthmaker semantics using a wider class of (non-canonical) models and provide equivalent characterizations of all our consequence relations. In this form of truthmaker semantics, we use structures that I call *R-spaces*, which are like ordinary state-spaces, but with a designated *reality* state, which we may think of as the fusion of all states in the space that obtain.

Definition 17 An *R-space* is any triple $(S, \sqsubseteq, r)$ with $(S, \sqsubseteq)$ a state-space and $r \in S$.

Definition 18 A bilateral proposition $P = (P^+, P^-)$ on an R-space $(S, \sqsubseteq, r)$ is

- *closed* iff P^+ and P^- are closed under non-empty fusions
- *decided* iff some element of $P^+ \cup P^-$ is part of r
- *consistent* iff no element of $P^+ \cap P^-$ is part of r
- *classical* iff decided and consistent

Proposition 11 *Closure, Decidedness, Consistency, and Classicality are preserved under conjunction, disjunction and negation as defined in definition 11.*

Proof Straightforward by application of the definitions. □

Definition 19

- An *R-model* is any quadruple $(S, \sqsubseteq, r, [\cdot])$ with $(S, \sqsubseteq, r)$ an R-space and $[\cdot]$ a function mapping each atom $p \in At$ to a closed bilateral proposition on $(S, \sqsubseteq)$.
- An R-model $(S, \sqsubseteq, r, [\cdot])$ is *complete* (*consistent*, *classical*) iff every value of $[\cdot]$ is decided (consistent, classical).

Given an R-model, $[\cdot]$ is extended to complex formulas in the same as for state-models in general (see definition 12). By proposition 11, in a complete (consistent, classical) model, $[A]$ is decided (consistent, classical) for all $A \in L$.

Definition 20 Let $\Gamma \cup \{C\} \subseteq$L and $M = (S, \sqsubseteq, r, [\cdot])$ an R-model. Then $\Gamma \Rightarrow^M C$ iff: if r inexactly verifies Γ, then the maximal part of r verifying Γ contains a verifier of C.

Definition 21 Let $\Gamma \cup \{C\} \subseteq L$. Then

- $\Gamma \models_{R4} C$ iff $\Gamma \Rightarrow^M C$ for every R-model M
- $\Gamma \models_{RI} C$ iff $\Gamma \Rightarrow^M C$ for every complete R-model M
- $\Gamma \models_{RP} C$ iff $\Gamma \Rightarrow^M C$ for every consistent R-model M
- $\Gamma \models_{R2} C$ iff $\Gamma \Rightarrow^M C$ for every classical R-model M

To prove equivalence to our web-consequence relations, we associate the reality state in an R-model with a corresponding canonical state.

Definition 22 For any R-model with reality state r, let $r^* = \{\phi \in Lit : r$ contains a verifier of $\phi\}$.

It is readily seen that whenever an R-model M with reality state r is complete (consistent, classical), r^* is also complete (consistent, classical).

Lemma 12 *For any R-model M, if $\Gamma \Rightarrow_{r^*} C$, also $\Gamma \Rightarrow^M C$.*

Proof In what follows, we use subscripts C and M for the model-relative terms such as 'verifies' and 'contains' to make clear if they are to be understood relative to the canonical model or the R-model M.

For any $\phi \in r^*$, let ϕ° be some part$_M$ of r verifying$_M$ ϕ, and for each $z \subseteq r^*$, let z° be $\bigsqcup\{\phi^\circ : \phi \in z\}$. Note that for any z, z_1, $z_2 \subseteq r^*$ we have that $z^\circ \sqsubseteq r$, that $z_1^\circ \sqcup z_2^\circ = (z_1 \cup z_2)^\circ$ and that $z_1^\circ \sqsubseteq z_2^\circ$ if $z_1 \subseteq z_2$. We first prove by induction that

(*) if r contains$_M$ a verifier$_M$ of A then r^* contains$_C$ a verifier$_C$ of A and, for all $z \subseteq r^*$, if z verifies$_C$ A, then z° verifies$_M$ A.

In the base case, A is a literal. But if r contains$_M$ a verifier$_M$ of literal ϕ, then by construction, r^* contains$_C$ $\{\phi\}$, which verifies$_C$ ϕ. If $z \subseteq r^*$ verifies$_C$ a literal ϕ, $z = \{\phi\}$. Then $z^\circ = \bigsqcup\{\phi^\circ\} = \phi^\circ$, which is a verifier$_M$ of ϕ, as required.

Now suppose (*) holds of A and B (IH). We prove that it then also holds of $(A \wedge B)$, $(A \vee B)$, $\neg(A \wedge B)$, $\neg(A \vee B)$, and $\neg\neg A$.

If r contains$_M$ a verifier$_M$ of $A \wedge B$, then r contains$_M$ a verifier$_M$ of A and a verifier$_M$ of B. By IH, r^* contains$_C$ a verifier$_C$ of A and a verifier$_C$ of B, and hence contains a verifier$_C$ of $A \wedge B$. If $z \subseteq r^*$ verifies$_C$ $A \wedge B$, then $z = z_1 \cup z_2$ with z_1, z_2

verifying$_C$ A, B, respectively. By IH, z_1° verifies$_M$ A and z_2° verifies$_M$ B. Then $z_1^\circ \sqcup z_2^\circ$ verifies$_M$ $A \wedge B$ and is equal to $(z_1 \cup z_2)^\circ = z^\circ$.

If r contains$_M$ a verifier$_M$ of $A \vee B$, then r contains$_M$ a verifier$_M$ of A or of B. By IH, r^* contains$_C$ a verifier$_C$ of A or of B, and hence contains$_C$ a verifier$_C$ of $A \vee B$. If $z \subseteq r^*$ verifies$_C$ $A \vee B$, either (i) z verifies$_C$ A, or (ii) z verifies$_C$ B, or (iii) $z = z_1 \cup z_2$ with z_1, z_2 verifying$_C$ A, B, respectively. If (i), by IH, z° verifies$_M$ A and hence $A \vee B$. Similarly if (ii). If (iii), the reasoning is as before.

The other cases are similar. This completes the proof of (*).

Now suppose $\Gamma \not\Rightarrow^M C$, so r inexactly verifies$_M$ Γ, but there are no $u \sqsubseteq t \sqsubseteq r$ with u verifying$_M$ C and t verifying$_M$ Γ. By (*), r^* inexactly verifies$_C$ Γ. Suppose for contradiction that $\Gamma \Rightarrow_{r^*} C$, so there are $x \subseteq y \subseteq r^*$ with x verifying$_C$ C and y verifying$_C$ Γ. Then $x^\circ \sqsubseteq y^\circ \sqsubseteq r$, and by (*), x° verifies$_M$ C and y° verifies$_M$ Γ, contrary to assumption. □

Theorem 13 *Let* $\Gamma \cup \{C\} \subseteq L$. *Then* $\Gamma \models_* C$ *iff* $\Gamma \models_{R*} C$ *for all* $* \in \{4, I, P, 2\}$

Proof We do the case of $* = 4$ and sketch how to adapt the reasoning to the other cases. We use subscripts for model-relative vocabulary as in the previous proof.

Left-to-right: Suppose $\Gamma \not\models_{R4} C$, so there is an R-model $M = (S, \sqsubseteq, r, [\cdot])$ such that $\Gamma \not\Rightarrow^M C$. By lemma 12, $\Gamma \not\Rightarrow_{r^*} C$. Since $r^* \in S^C$, it follows that $\Gamma \not\models_{C4} C$, and hence by proposition 7, that $\Gamma \not\models_4 C$. The reasoning generalizes to the other cases, using the fact that r^* is a complete (consistent, classical) canonical state if the R-model M is complete (consistent, classical).

Right-to-left: Suppose $\Gamma \not\models_4 C$. By proposition 7, $\Gamma \not\models_{C4} C$, so in M^C, there is an $s \in S^C$ such that s inexactly verifies$_C$ Γ, but it is not the case that the maximal part of s verifying$_C$ Γ contains$_C$ a verifier$_C$ of C. Since $(S^C, \sqsubseteq^C, s, [\cdot]^C)$ is an R-model, it follows that $\Gamma \not\models_{R4} C$. The reasoning generalizes to the other cases, using the fact that the R-model $(S^C, \sqsubseteq^C, s, [\cdot]^C)$ is complete (consistent, classical) if s is complete (consistent, classical). □

5 Many-Valued Semantics

In this section, I present many-valued characterizations of the consequence relations defined in the previous sections. To see how this works, it helps to first bring the previous definitions into a slightly different shape, more similar to standard definitions of consequence in terms of the preservation of some property from premises to conclusion (sec. 5.1), before specifying the many-valued system itself (sec. 5.2) and using it to define consequence relations provably equivalent to the web consequence relations (sec. 5.3).

5.1 Consequence as Preservation

As part of the informal introduction of his proposal, Schnieder suggests that we may think of it as replacing 'the idea of truth-preservation, which is central to all standard accounts of consequence, with the idea of *ground-preservation*' (**?**, p. S1346). It turns out to be instructive to pursue this suggestion. As Schnieder emphasizes, most familiar definitions of consequence follow a common pattern: they generalize over a class of models, or interpretations, of the language in question, and then ask whether the conclusion is true under a given interpretation, provided that each premise is true under that same interpretation. Different conceptions of consequence then result from different conceptions of the range of interpretations, and of what it is for a formula to be true under an interpretation.

Schnieder's remark suggests that his definition of consequence instantiates a similar pattern, but with a ground-theoretic property taking the place of the property of truth. At first glance, however, it seems that neither Schnieder's own definitions nor our various reformulations and variations really fit this pattern. For present purposes, the point is best illustrated by the following characterization:

- $\Gamma \models_{4/I/P/2} C$ iff for any arbitrary / complete / consistent / classical state $s \in S^C$, if s contains a verifier of Γ, then the maximal verifier of Γ contained in s contains a verifier of C.

Here, the generalization over a given class of states plays the role of a generalization over interpretations. Then with respect to any such interpretation, we ask if the premise set has a certain property: being verified by part of the interpretation-state—and given the close connection between verification and grounding, we may reasonably count this as a ground-theoretic property. But if the premise set has that property, we do *not* go on to ask whether the conclusion *also* has that property with respect to the interpretation. Instead, we ask whether the conclusion stands in a certain ground-theoretic *relation to the premises*, relative to the interpretation being considered.

It turns out, however, that these definitions can be reformulated in a way that is closer to the standard pattern. The key is to regard as interpretations not simply the (complete, consistent, classical) states generalized over above, but *pairs* of such states with designated parts, or equivalently, states paired with arbitrary (complete, consistent, classical) extensions. To make this precise, let us say that a state s is an s^+-maximal verifier of a formula A iff s verifies A, s is part of s^+, and s is the maximal verifier of A that is part of s^+, and define the following three 'locality-grades' of verification of a formula by such a pair of states, capturing to what extent verification of the formula by the pair (s, s^+) is confined to the designated part s:

Definition 23 Let $A \in \Gamma$ and $s \sqsubseteq s^+ \in S^C$

- (s, s^+) *verifies* A iff s^+ contains a verifier of A
- (s, s^+) *locally verifies* A iff s contains a verifier of A
- (s, s^+) *superlocally verifies* A iff s contains an s^+-maximal verifier of A

Is there an informal gloss on these two-component interpretations (s, s^+)? It may be helpful here to recall Schnieder's informal gloss: '[a] proposition is a consequence of some premises if, given that they are grounded themselves, they *provide* a ground of it or *bring* such a ground *along*' ((**?**, p. S1346)). Very roughly, in an interpretation (s, s^+), one can think of s^+ as the overall situation relative to which we are evaluating given premises Γ and conclusion C, and s as that part of the overall situation we consider as putatively 'brought along' by the premise set. Thus, we have a counter-model if Γ is grounded in s^+, if the designated part s comprises the entirety of what Γ brings along in s^+, but s, and hence what Γ brings along in s^+, does not include a ground of the conclusion.

We can now establish the following characterization of our four consequence relations:

Proposition 14 $\Gamma \models_{4/I/P/2} C$ *iff*
for every state s and arbitrary / complete / consistent / classical extension s^+, if (s, s^+) superlocally verifies each $A \in \Gamma$, then (s, s^+) locally verifies C.

Proof We give a proof for the case of $\models_4$ and a state s paired with an arbitrary extension s^+; it carries over to the other cases straightforwardly.

Left-to-right: Assume $\Gamma \models_4 P$. Then by proposition 7 and lemma 6, for any state s containing a verifier of Γ, the s-maximal verifier of Γ contains a verifier of C. Now let $s \sqsubseteq s^+$ and suppose (s, s^+) superlocally verifies each $A \in \Gamma$. It follows that s^+ contains a verifier of Γ, and that s contains the s^+-maximal verifier of Γ. Call that state t. By assumption, t contains a verifier of C, and hence so does s. So (s, s^+) locally verifies C, as required.

Right-to-left: Assume that whenever $s \sqsubseteq s^+$, if (s, s^+) superlocally verifies each $A \in \Gamma$, then (s, s^+) locally verifies C. Let s^+ be any state containing a verifier of Γ, and let s be the s^+-maximal verifier of Γ. We need to show that s contains a verifier of C. Since s is the s^+-maximal verifier of Γ, s contains an s^+-maximal verifier of each $A \in \Gamma$, so (s, s^+) superlocally verifies each $A \in \Gamma$. By assumption, (s, s^+) locally verifies C, and hence s contains a verifier of C, as required. $\square$

There are two features of the conditions here identified as necessary and sufficient for our consequence relations to obtain that require comment. First, in contrast to the previous characterizations of the relations, the property that we are appealing to with respect to the conclusion—i.e. being locally verified—is one it has, or lacks, relative only to the interpretation, rather than relative to the interpretation and the given premise set. This makes it possible to encode the property by means of a value within a many-valued semantics, as we shall see in the next section.

Second, while the shape of the conditions is reminiscent of definitions of consequence as some form of truth-preservation, the condition is not actually one of preservation of one and the same ground-theoretic property—it is required only that if all premises are superlocally verified, then the conclusion is locally verified. We might therefore regard the relation as one of partial or imperfect preservation: if the premises are verified with the highest degree of locality, the conclusion is verified at least with the second-highest degree of locality. In this way, the web consequence relations are very similar to the consequence relations studied in so-called *Strict/Tolerant* logic, which is based precisely on the idea that the premises in an inference are to be held to higher standards—need to be true in a stronger or stricter sense—than the conclusion.[13] Since they require only a form of imperfect preservation of a property from premises to conclusion, strict/tolerant logics are typically *non-transitive*. Unsurprisingly, the same is true of two of our web consequence relations, namely $\models_I$ and $\models_2$. The same example serves to establish the non-transitivity of both consequence relations. It is easily verified that we have $P \models_I P \vee (Q \vee \neg Q)$ and $P \vee (Q \vee \neg Q) \models_I Q \vee \neg Q$, but $P \not\models_2 Q \vee \neg Q$. Since every $\models_I$-entailment is a $\models_2$-entailment, it follows that both $\models_I$ and $\models_2$ are non-transitive. The other two web consequence relations $\models_4$ and $\models_P$ are of course transitive, since they coincide with FDE and K_3, respectively, for which we gave definitions in terms of preservation of inexact verification above.

5.2 Values

In terms of our three locality-grades of verification, relative to any pair of a state s and an extension s^+, we can sort the formulas into four mutually exclusive and exhaustive categories: not verified by (s, s^+), verified but not locally verified by (s, s^+), locally but not superlocally verified by (s, s^+), and superlocally verified by (s, s^+). The key idea behind the many-valued system I wish to propose is to represent each of these four categories by a separate value. And the key fact which makes this possible is that, as we show below, which of the categories a conjunction or disjunction belongs to is a function of which of the categories its conjuncts or disjuncts belong to. The same is not true for negation: we cannot in general tell from the category of A which category $\neg A$ belongs to. So we assign to each formula a pair of values, the first representing the category of the formula itself, and the second representing the category of its negation. Let us make this precise.

Definition 24

- A *value* is any element of $V = \{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$
- A value in V is
  - *complete* iff at most one of its coordinates is 0,
  - *consistent* iff at least one of its coordinates is 0,
  - *classical* iff consistent and complete.
- A *web-valuation* is any function $v : At \to V$

[13] On the idea of *strict/tolerant* logics, see e.g. **????**. **?** proposes a truthmaker semantics for strict/tolerant logic, which is quite different from the ones considered here, though. It would be interesting to study the relationship between web consequence and strict/tolerant logic in more detail, but in this paper, we shall limit ourselves to the observation that standard strict/tolerant logic, in terms of its consequence relation, coincides with classical logic, and thus differs from all of the web consequence relations.

- A web-valuation v is *complete (consistent, classical)* iff for all $p \in At$, $v(p)$ is complete (consistent, classical).

For $A \in L$ and v a web-valuation, we write $v^+(A)$ for the first and $v^-(A)$ for the second coordinate of $v(A)$. Informally, $v^+(A) = 0(1, 2, 3)$ represents that A is not (merely, merely locally, superlocally) verified by the interpretation encoded in v, and $v^-(A) = 0(1, 2, 3)$ represents that $\neg A$ is not (merely, merely locally, superlocally) verified by the interpretation.

To extend a web-valuation v to complex formulas, we first define binary conjunctive and disjunctive functions on the set $\{0, 1, 2, 3\}$ by the following tables. The conjunctive function is then used to determine the positive value of a conjunction and the negative value of a disjunction, and dually, the disjunctive function is used to determine the positive value of a disjunction and the negative value of a conjunction.

c	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

d	0	1	2	3
0	0	1	2	3
1	1	1	2	**2**
2	2	2	2	**2**
3	3	**2**	**2**	3

Note that the c-function is simply the min-function, as might be expected for conjunction. The d-function is close to the max-function, but diverges from it in the highlighted cases, when one argument is 3 and the other 1 or 2, in which case it yields the value 2. This is justified by the informal interpretation: if A is superlocally verified and B is verified but not superlocally verified by (s, s^+), then some verifier of A and hence $A \vee B$ is contained in s, so $A \vee B$ is locally verified, but B and hence $A \vee B$ is verified by a part of s^+ not contained in s, so $A \vee B$ is not superlocally verified.

Definition 25 For v a valuation and $A, B \in L$, we let

- $v(\neg A) = (v^-(A), v^+(A))$
- $v(A \wedge B) = (c(v^+(A), v^+(B)), d(v^-(A), v^-(B))$
- $v(A \vee B) = (d(v^+(A), v^+(B)), c(v^-(A), v^-(B))$

Proposition 15 *If a web-valuation v is complete (consistent, classical), then for all $A \in L$, $v(A)$ is complete (consistent, classical).*

Proof By induction. The case of atoms is immediate. So suppose $v(A)$ and $v(B)$ are complete (IH). Then obviously, so is $v(\neg A)$. For $A \wedge B$, suppose that $v^-(A \wedge B) = 0$. By the table for the d-function, $v^-(A) = v^-(B) = 0$. By IH, neither of $v^+(A)$ and $v^+(B)$ is 0. By the table for the c-function, $v^+(A \wedge B)$ is also not 0, so $v(A \wedge B)$ is complete. The case of $v(A \vee B)$ is dual. Suppose now that $v(A)$ and $v(B)$ are consistent (IH). Then obviously, so is $v(\neg A)$. For $A \wedge B$, suppose $v^+(A \wedge B) \neq 0$. By the table for the c-function, neither of $v^+(A)$ and $v^+(B)$ is 0. By IH, $v^+(A) = v^+(B) = 0$. By the table for the d-function, $v^-(A \wedge B) = 0$, so $v(A \wedge B)$ is consistent. The case of $A \vee B$ is dual. The case of classical valuations then follows straightforwardly. □

5.3 Consequence

We will now define four consequence relations within the many-valued setting and prove them equivalent to the four web consequence relations.

Definition 26 Let $\Gamma \cup \{C\} \subseteq L$. Then

- $\Gamma \models_{M4} C$ iff for every web-valuation v, if $v^+(A) = 3$ for all $A \in \Gamma$, then $v^+(C) \geq 2$.
- $\Gamma \models_{MI} C$ iff for every complete web-valuation v, if $v^+(A) = 3$ for all $A \in \Gamma$, then $v^+(C) \geq 2$.
- $\Gamma \models_{MP} C$ iff for every consistent web-valuation v, if $v^+(A) = 3$ for all $A \in \Gamma$, then $v^+(C) \geq 2$.
- $\Gamma \models_{M2} C$ iff for every classical web-valuation v, if $v^+(A) = 3$ for all $A \in \Gamma$, then $v^+(C) \geq 2$.

To prove equivalence, we would like first to associate any web-valuation v with a corresponding pair (s, s^+) from our canonical state-space in such a way that whenever $v^+(A) = 3$, A is superlocally verified by (s, s^+), whenever $v^+(A) = 2$, then A is merely locally verified by (s, s^+), and so on. Given our definition of a canonical model, however, this is not always possible. For instance, there are no atoms p and pairs (s, s^+) such that p is locally, but not superlocally verified by (s, s^+). The reason is that atoms have only one verifier, which is automatically their maximal verifier. Fortunately, it will turn out that we can safely ignore web-valuations for which this problem arises, i.e. web-valuations that assign a positive or negative value of 2 to some atom.[14]

Definition 27

- A web-valuation is called *simple* if it never assigns an atom a positive or negative value of 2.
- For v a simple web-valuation,
  - $s(v) := \{\phi \in Lit : v^+(\phi) = 3\}$
  - $s^+(v) := \{\phi \in Lit : v^+(\phi) > 0\}$
- For v a web-valuation, the simplification v_s of v is just like v except that $v_s^+(p) = 1$ whenever $v^+(p) = 2$ and $v_s^-(p) = 1$ whenever $v^-(p) = 2$.

[14]Note that this does not mean that the value of 2 is entirely dispensable. Indeed, we will still have valuations that assign a positive or negative value of 2 to some complex formulas, for instance when $v^+(p) = 3$ and $v^+(q) = 1$ and therefore $v^+(p \vee q) = 2$.

Note that the definitions of $s(v)$ and $s^+(v)$ ensure that for all simple valuations v, $s^+(v)$ is complete (consistent, classical) if v is, and always $s(v) \sqsubseteq s^+(v)$.

Lemma 16 *For all $A \in L$,*

1. *if $v^+(A) = 3$, then A is superlocally verified by $(s(v), s^+(v))$,*
2. *if $v^+(A) < 2$, then A is not locally verified by $(s(v), s^+(v))$,*
3. *if $v^+(A) = 0$, then A is not verified by $(s(v), s^+(v))$*

Proof For the case of A a literal, suppose $\phi \in Lit$. Note that in the canonical model, $\{\phi\}$ is the sole and therefore maximal verifier of ϕ. If $v^+(\phi) = 3$, then by definition, $\phi \in s(v)$, so $s(v)$ contains the $s^+(v)$-maximal verifier of ϕ, and hence ϕ is superlocally verified by $(s(v), s^+(v))$. If $v^+(\phi) < 2$, then by definition, $\phi \notin s(v)$, so $s(v)$ contains no verifier of ϕ, and hence ϕ is not locally verified by $(s(v), s^+(v))$. If $v^+(\phi) = 0$, then by definition, $\phi \notin s^+(v)$, so $s^+(v)$ contains no verifier of ϕ, and hence ϕ is not verified by $(s(v), s^+(v))$.

Now suppose for induction that (IH) A and B satisfy the claim. We can then show that so do $(A \wedge B)$, $(A \vee B)$, $\neg(A \wedge B)$, and $\neg(A \vee B)$. We give the proof for the cases of $(A \wedge B)$ and $(A \vee B)$, the other two cases are dual.

If $v^+(A \wedge B) = 3$, then $v^+(A) = v^+(B) = 3$, so by (IH), both A and B are superlocally verified by $(s(v), s^+(v))$. Then $s(v)$ contains the fusion of the $s^+(v)$-maximal verifier of A and the $s^+(v)$-maximal verifier of B, which is the $s^+(v)$-maximal verifier of $(A \wedge B)$, hence $(A \wedge B)$ is superlocally verified by $(s(v), s^+(v))$. For the case of $v^+(A \wedge B) < 2$, the key is to observe that $v^+(A \wedge B) < 2$ only if either $v^+(A) < 2$ or $v^+(B) < 2$, and that $A \wedge B$ is locally verified by $(s(v), s^+(v))$ only if both A and B are. The case of $v^+(A \wedge B) = 0$ is analogous.

Now suppose $v^+(A \vee B) = 3$. There are two cases. In the first, $v^+(A) = v^+(B) = 3$. Then the reasoning is as in the case of conjunction above, since the fusion of the $s^+(v)$-maximal verifiers of A and B is also the $s^+(v)$-maximal verifier of $(A \vee B)$. In the second case, one of $v^+(A)$ and $v^+(B)$ is 3, and the other 0. Without loss of generality, assume $v^+(A) = 3$ and $v^+(B) = 0$. Then by IH, A is superlocally verified by $(s(v), s^+(v))$ and B is not verified by $(s(v), s^+(v))$, i.e. $s^+(v)$ does not contain a verifier of B. Then the $s^+(v)$-maximal verifier of A is also the $s^+(v)$-maximal verifier of $(A \vee B)$, and hence $(A \vee B)$ is superlocally verified $(s(v), s^+(v))$, as required. □

Lemma 17 *For v any valuation and $A \in L$:*

1. *If $v^+(A) = 3$ then $v_s^+(A) = 3$ and if $v^-(A) = 3$ then $v_s^-(A) = 3$*
2. *If $v^+(A) < 2$ then $v_s^+(A) < 2$ and if $v^-(A) < 2$ then $v_s^-(A) < 2$*
3. *If $v^+(A) = 0$ then $v_s^+(A) = 0$ and if $v^-(A) = 0$ then $v_s^-(A) = 0$*

Proof By a straightforward induction using the tables for c and d and the definition of simplification. □

Next, we associate any pair $s \sqsubseteq s^+$ from our canonical state-space with a corresponding simple valuation.

Definition 28 For any canonical states $s \sqsubseteq s^+$, let $v_{(s,s^+)}$ be such that $v^+_{(s,s^+)}(p)$ is 3 if $p \in s$, 1 if $p \in s^+$ but $p \notin s$, and 0 otherwise, and $v^-_{(s,s^+)}(p)$ is 3 if $\neg p \in s$, 1 if $\neg p \in s^+$ but $\neg p \notin s$, and 0 otherwise.

Note that $v_{(s,s^+)}$ is complete (consistent, classical) if s^+ is.

Lemma 18 *For s, s^+ canonical states with $s \sqsubseteq s^+$ and $A \in L$:*

1. *If (s, s^+) superlocally verifies A, then $v^+_{(s,s^+)}(A) = 3$*
2. *If (s, s^+) does not locally verify A, then $v^+_{(s,s^+)}(A) < 2$*
3. *If (s, s^+) does not verify A, then $v^+_{(s,s^+)}(A) = 0$*

Proof For the case of A a literal, suppose first that (s, s^+) superlocally verifies ϕ, so $\phi \in s$. Then by definition, if ϕ is an atom, $v^+_{(s,s^+)}(\phi) = 3$, and if ϕ is the negation of an atom p, $v^-_{(s,s^+)}(p) = 3$ and so again $v^+_{(s,s^+)}(\phi) = 3$. If (s, s^+) does not locally verify ϕ, then $\phi \notin s$. Similarly as before, by definition, whether ϕ is an atom or the negation of an atom, $v^+_{(s,s^+)}(\phi)$ is either 0 or 1, and hence < 2. If (s, s^+) does not locally verify ϕ, then $\phi \notin s^+$, and by definition, $v^+_{(s,s^+)}(\phi) = 0$.

Now suppose for induction that (IH) A and B satisfy the claim. We can then show that so do $(A \wedge B)$, $(A \vee B)$, $\neg(A \wedge B)$, and $\neg(A \vee B)$. For illustration, we give the proof for the case of $(A \vee B)$, the other cases follow a similar pattern. So suppose first that (s, s^+) superlocally verifies $A \vee B$, so the s^+-maximal verifier of $A \vee B$ is contained in s. There are two cases. In the first, the s^+-maximal verifier of $A \vee B$ is the fusion of s^+-maximal verifiers of A and B, in which case A and B are both superlocally verified by (s, s^+). Then by IH, $v^+_{(s,s^+)}(A) = 3$ and $v^+_{(s,s^+)}(B) = 3$, so $v^+_{(s,s^+)}(A \vee B) = 3$. In the second case, the s^+-maximal verifier of $A \vee B$ is the s^+-maximal verifiers of either A or B, with s not containing any verifier of the other one of A and B. In that case, one of A and B is superlocally verified by (s, s^+) and the other not even locally verified. Then by IH, one has positive value 3 and the other 0, so $v^+_{(s,s^+)}(A \vee B) = 3$, as required. Suppose next that (s, s^+) does not locally verify $A \vee B$. Then s contains no verifier of A and no verifier of B, and hence (s, s^+) does not locally verify either. So by IH, $v^+_{(s,s^+)}(A) < 2$ and $v^+_{(s,s^+)}(B) < 2$ and hence $v^+_{(s,s^+)}(A \vee B) < 2$. Suppose finally that (s, s^+) does not verify $A \vee B$. Then s^+ contains no verifier of A and no verifier of B, and hence (s, s^+) does not verify either. So by IH, $v^+_{(s,s^+)}(A) = 0$ and $v^+_{(s,s^+)}(B) = 0$ and hence $v^+_{(s,s^+)}(A \vee B) = 0$. □

Theorem 19 *Let $\Gamma \cup \{C\} \subseteq L$. Then $\Gamma \models_* C$ iff $\Gamma \models_{M*} C$ for all $* \in \{4, I, P, 2\}$*

Proof We give the proof for $\models_4$. Left-to-right: Suppose there is a web-valuation v with $v^+(A) = 3$ for all $A \in \Gamma$ and $v^+(C) < 2$. By lemma 17, $v^+_s(A) = 3$ for all $A \in \Gamma$ and $v^+_s(C) < 2$. By lemma 16, every $A \in \Gamma$ is superlocally verified by $(s(v_s), s^+(v_s))$, but C is not locally verified by $(s(v_s), s^+(v_s))$. By proposition 14, $\Gamma \not\models_4 C$. The reasoning straightforwardly carries over to the other cases, using our observations that simplification preserves completeness and consistency of a valuation, and that for simple v, $s^+(v)$ is complete (consistent, classical) if v is.

Right-to-left: suppose $\Gamma \not\models_4 C$. By proposition 14, in the canonical model there are states $s \sqsubseteq s^+$ with (s, s^+) superlocally verifying each $A \in \Gamma$ but not locally verifying C. By lemma 18, $v^+_{(s,s^+)}(A) = 3$ for all $A \in \Gamma$ but $v^+_{(s,s^+)}(C) < 2$, as required. The reasoning again carries over to the other cases, using the observation that $v_{(s,s^+)}$ is complete (consistent, classical) if s^+ is. □

6 Tableaux

In this section, I describe tableaux-based proof procedures which are sound and complete for $\models_4$, $\models_I$, $\models_P$, and $\models_2$, respectively. The tableaux generated are the same in each case, with the different derivability relations resulting from different definitions of what it is for a tableau to be closed. I will make use of a kind of tableaux similar to those employed in (**?**, esp. ch. 8) for logics such as FDE, placing formulas paired with certain flags on the nodes. Informally, the flags represent conditions on the positive value of the relevant formula. We shall therefore use as flags certain subsets of the set V. A node $A, \{0, 1\}$ is then to be read as representing the condition that formula A has a positive value of either 0 or 1. As usual, the construction of a tableau testing an inference from Γ to C represents a systematic and exhaustive search for a counter-model to the entailment of C by Γ, i.e. a valuation in which each premise has positive value 3, but in which the conclusion has a positive value less than 2.

We will need to use three flags—$\{3\}$, $\{0, 1\}$, and $\{0\}$—which we abbreviate as 3, <2, and 0, respectively. For each flag, we give rules for extending tableaux with a leaf node with that flag. We begin with the rules for 3-flagged formulas.

(3:¬¬)

$$\begin{array}{c} \neg\neg A,\ 3 \\ | \\ A,\ 3 \end{array}$$

(3:∧)

$$\begin{array}{c} A \wedge B,\ 3 \\ | \\ A,\ 3 \\ B,\ 3 \end{array}$$

(3:¬∧)

$$\begin{array}{ccc} & \neg(A \wedge B),\ 3 & \\ \swarrow & \downarrow & \searrow \\ \neg A,\ 3 & \neg A,\ 3 & \neg A,\ 0 \\ \neg B,\ 3 & \neg B,\ 0 & \neg B,\ 3 \end{array}$$

(3:∨)

$$\begin{array}{ccc} & A \vee B,\ 3 & \\ \swarrow & \downarrow & \searrow \\ A,\ 3 & A,\ 3 & A,\ 0 \\ B,\ 3 & B,\ 0 & B,\ 3 \end{array}$$

(3:¬∨)

$$\begin{array}{c} \neg(A \vee B),\ 3 \\ | \\ \neg A,\ 3 \\ \neg B,\ 3 \end{array}$$

With the exception of the rules for disjunctions and negated conjunctions, the rules are exactly like the rules for +-signed formulas in Priest's tableau calculus for FDE. The exceptions correspond to the 'non-standard' entries in the disjunction-table discussed above: a disjunction has value 3 whenever one disjunct has value 3, *provided* the other one does not have value 1 or 2, in which case the disjunction is verified but not superlocally verified.

The rules for <2-flagged formulas are even more straightforward; in fact, they are exactly like Priest's rules for $-$-signed formulas.

(<2:¬¬)

$$\begin{array}{c} \neg\neg A, \ <2 \\ | \\ A, \ <2 \end{array}$$

(<2:∧)

$$\begin{array}{cc} \multicolumn{2}{c}{A \wedge B, \ <2} \\ / & \backslash \\ A, \ <2 & B, \ <2 \end{array}$$

(<2:¬∧)

$$\begin{array}{c} \neg(A \wedge B), \ <2 \\ | \\ \neg A, \ <2 \\ \neg B, \ <2 \end{array}$$

(<2:∨)

$$\begin{array}{c} A \vee B, \ <2 \\ | \\ A, \ <2 \\ B, \ <2 \end{array}$$

(<2:¬∨)

$$\begin{array}{cc} \multicolumn{2}{c}{\neg(A \vee B), \ <2} \\ / & \backslash \\ \neg A, \ <2 & \neg B, \ <2 \end{array}$$

The rules for 0-flagged formulas are exactly analogous to those for <2-flagged formulas, giving rise to new 0-flagged nodes instead of <2-flagged ones.

We now define various conditions under which a branch will be considered closed, depending on which consequence relation we are trying to find a counter-model for.

Definition 29 Let b be a branch in some given tableau. Then b is

- *gappy* iff for some formula A, both $A, \ 0$ and $\neg A, \ 0$ are on b
- *glutty* iff for some formula A, both $A, \ 3$ and $\neg A, \ 3$ are on b
- *4-closed* iff for some formula A, both $A, \ 3$ and $A, \ <2$ or $A, \ 0$ are on b
- *i-closed* iff 4-closed or gappy
- *p-closed* iff 4-closed or glutty
- *2-closed* iff 4-closed or gappy or glutty

We are now in a position to define provability from some premises. For simplicity, we shall restrict attention in this paper to finite sets of premises. There is a natural way of extending our treatment to the case of infinite premise sets, and the soundness and completeness results then carry over to the general case as well.[15]

[15] This may be done, for example, by a straightforward adaptation of the methods used in (**?**, sections 12.7-10).

Definition 30 Let $\Gamma \cup \{C\}$ be a finite subset of L.

- A tableau T *tests* $(\Gamma,\ C)$ iff T begins with $A,\ 3$ for all $A \in \Gamma$, followed by $C,\ < 2$.
- $\Gamma \vdash_4 C$ $(\Gamma \vdash_I C, \Gamma \vdash_P C, \Gamma \vdash_2 C)$ iff there is a tableau testing $(\Gamma,\ C)$ every branch of which is 4-closed (i-closed, p-closed, 2-closed)

We look at some simple examples of tableaux. We shall mark the fact that a certain branch satisfies one of the conditions for being closed by writing $\otimes$ below the last node, adding as a superscript an indication of the most demanding closure condition(s) the branch satisfies. Here is a very simple example, testing the inference from $A \wedge B$ to $B \vee C$.

$$\begin{array}{c} A \wedge B,\ 3 \\ B \vee C,\ \ {<}2 \\ B,\ \ {<}2 \\ C,\ \ {<}2 \\ A,\ 3 \\ B,\ 3 \\ \otimes^4 \end{array}$$

The one and only branch is 4-closed since it contains B both 3- and $<$ 2-flagged. So the tableau shows that the inference tested is valid in all four deductive systems under consideration.

Here is a slightly more complicated complete tableau testing the inference from $A \vee (B \vee \neg B)$ to $B \vee \neg B$ which illustrates the use of the somewhat unusual-looking rule for 3-flagged disjunctions.

$$\begin{array}{c} A \vee (B \vee \neg B),\ 3 \\ B \vee \neg B,\ \ {<}2 \\ B,\ \ {<}2 \\ \neg B,\ \ {<}2 \end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} A,\ 3 \\ B \vee \neg B,\ 3 \\ \begin{array}{ccc} B,\ 3 & B,\ 3 & B,\ 0 \\ \neg B,\ 3 & \neg B,\ 0 & \neg B,\ 3 \\ \otimes^4 & \otimes^4 & \otimes^4 \end{array} \end{array}
&
\begin{array}{c} A,\ 3 \\ B \vee \neg B,\ 0 \\ B,\ 0 \\ \neg B,\ 0 \\ \otimes^i \end{array}
&
\begin{array}{c} A,\ 0 \\ B \vee \neg B,\ 3 \\ \begin{array}{ccc} B,\ 3 & B,\ 3 & B,\ 0 \\ \neg B,\ 3 & \neg B,\ 0 & \neg B,\ 3 \\ \otimes^4 & \otimes^4 & \otimes^4 \end{array} \end{array}
\end{array}$$

The first three branches from the left are 4-closed because on each of them, either B or $\neg B$ occurs both 3-flagged and $<$ 2-flagged. The last three branches are also 4-closed for the same reason. The center branch is not 4-closed, but gappy—both B and $\neg B$ occur 0-flagged—and hence i-closed and 2-closed. So the tableau, since it is complete, shows at once that the inference being tested is valid in $\vdash_I$ and $\vdash_2$ but not in $\vdash_4$ and $\vdash_P$.

6.1 Soundness

Definition 31 A web-valuation v *validates* a node A, V' iff $v^+(A) \in V'$.

Definition 32 A tableau rule is sound iff for every node A, V' to which the rule applies, if a valuation v validates A, V', the rule produces some branch such that v validates every new node on the branch.

Lemma 20 *All of the web-tableau rules are sound.*

Proof We restrict ourselves to a few illustrative cases.

(3:$\vee$): Suppose the rule is applied to a node $A \vee B, 3$. It then produces three new branches. The first has the additional nodes $A, 3$ and $B, 3$, the second has the additional nodes $A, 3$ and $B, 0$, and the third has the additional nodes $A, 0$ and $B, 3$. Let v be a valuation validating $A \vee B, 3$, so $v^+(A \vee B) = 3$. Then by the disjunction table, either $v^+(A) = v^+(B) = 3$, or $v^+(A) = 3$ and $v^+(B) = 0$, or $v^+(A) = 0$ and $v^+(B) = 3$. In the first case, v validates every new node on the first new branch, in the second case, v validates every new node on the second new branch, and in the third case, v validates every new node on the third new branch.

(<2:$\wedge$): Suppose the rule is applied to a node $A \wedge B, < 2$. It then produces two new branches. The first has the additional node $A, < 2$, and the second has the new node $B, < 2$. Let v be a valuation validating $A \vee B, < 2$. Then by the conjunction table, either $v^+(A) < 2$ or $v^+(B) < 2$. In the first case, v validates the new node on the first branch, and in the second case, v validates the new node on the second branch. □

Definition 33 A web-valuation v is *faithful* to a branch b iff v validates every node on b.

Lemma 21 *Let v be web-valuation which is faithful to branch b of some tableau. Then*

1. *b is not 4-closed*
2. *If v is complete, b is not gappy*
3. *If v is consistent, b is not glutty*

Proof Let v be a web-valuation faithful to branch b.

For 1.: Suppose b is 4-closed. Then for some formula A, both $A, 3$ and $A, < 2$ are on b. Since v is faithful to b, v validates both $A, 3$ and $A, < 2$, so $v^+(A) = 3$ and $v^+(A) < 2$. Contradiction. So b is not 4-closed.

For 2.: Suppose b is gappy. Then for some formula A, both $A, 0$ and $\neg A, 0$ are on b. Since v is faithful to b, v validates both $A, 0$ and $\neg A, 0$, so $v^+(A) = 0$ and $v^-(A) = 0$. By proposition 15, it follows that v is not complete.

For 3.: Suppose b is glutty. Then for some formula A, both $A, 3$ and $\neg A, 3$ are on b. Since v is faithful to b, v validates both $A, 0$ and $\neg A, 0$, so $v^+(A) = 3$ and $v^-(A) = 3$. By proposition 15, it follows that v is not consistent. □

Theorem 22 *Let $\Gamma \cup \{C\}$ be a finite subset of L. Then $\Gamma \vdash_* C$ implies $\Gamma \models_* C$ for all $* \in \{4, I, P, 2\}$*

Proof Let $\Gamma \cup \{C\}$ be a finite subset of L. For the case of $* = 4$, suppose $\Gamma \not\models_4 C$. Then there is a valuation v with $v^+(A) = 3$ for all $A \in \Gamma$ and $v^+(C) < 2$. Suppose for contradiction that $\Gamma \vdash_4 C$, so there is a tableau T testing (Γ, C) every branch of which is 4-closed. T begins with $A, 3$ for all $A \in \Gamma$, followed by $C, < 2$, so v is faithful to this initial segment T_0 of T. Since T is obtained by extending T_0 through application of the rules, and since all the rules are sound, v is faithful to some branch b of T. By lemma 21, this contradicts the assumption that every branch of T is 4-closed. The other cases are parallel. □

6.2 Completeness

We now prove completeness of the tableau procedures with respect to the corresponding many-valued consequence relations. First, we show that each tableau rule is complete in the following sense:

Definition 34 A tableau rule is *complete* iff whenever a valuation v validates every new node on some branch produced in applying the rule to a node A, V', then v also validates A, V'.

Lemma 23 *All of the web-tableau rules are complete.*

Proof We again restrict ourselves to a few illustrative cases.

(3:$\vee$) The rule applies to any node $A \vee B, 3$ and then produces three branches b_1, b_2, b_3 whose new nodes are as follows. b_1: $A, 3$ and $B, 3$; b_2: $A, 3$ and $B, 0$; b_3: $A, 0$ and $B, 3$. It is immediate from the positive table for $\vee$ that if $v^+(A) = v^+(B) = 3$, or $v^+(A) = 3$ and $v^+(B) = 0$, or $v^+(A) = 0$ and $v^+(B) = 3$, then $v^+(A \vee B) = 3$, as required.

(<2:$\wedge$) The rule applies to any node $A \wedge B, < 2$ and then produces two branches b_1, b_2 whose new nodes are as follows. b_1: $A, < 2$; b_2: $B, < 2$. It is immediate from the positive table for $\wedge$ that if $v^+(A) < 2$ or $v^+(B) < 2$ then $v^+(A \wedge B) < 2$, as required. □

We then read off suitable valuations from complete open branches.

Definition 35 Let b be any complete 4-open branch. For $\phi \in Lit$, let $C_b(\phi)$ be V if no node of the form ϕ, V' occurs on b, and $\bigcap\{V' \subset V : \phi, V' \text{ is on } b\}$ otherwise.

Proposition 24 *If b is a complete 4-open branch and $\phi \in Lit$, $C_b(\phi)$ is non-empty.*

Proof Obviously, $C_b(\phi)$ is non-empty if no node of the form ϕ, V' occurs on b, so suppose some nodes of that form occur on b. Since b is 4-open, if $\phi, 3$ occurs on b, then neither $\phi, < 2$ nor $\phi, 0$ occurs on b, so $C_p(\phi) = \{3\}$. And if $\phi, 3$ does not occur on b, then clearly $0 \in C_b(\phi)$. So either way, $C_b(\phi)$ is non-empty. □

Definition 36 Let b be any complete 4-open branch.

- The *max-valuation induced* by b is the valuation $v_{b,max}$ such that
  - $v^+_{b,max}(p) = max(C_b(p))$
  - $v^-_{b,max}(p) = max(C_b(\neg p))$
- The *min-valuation induced* by b is the valuation $v_{b,min}$ with
  - $v^+_{b,min}(p) = min(C_b(p))$
  - $v^-_{b,min}(p) = min(C_b(\neg p))$

Definition 37 For valuations v_0, v_1, let $v_0 \leq v_1$ iff for all $p \in At$, $v^+_0(p) \leq v^+_1(p)$ and $v^-_0(p) \leq v^-_1(p)$

Lemma 25 *Let b be any complete 4-open branch. Then every valuation v with $v_{b,min} \leq v \leq v_{b,max}$ is faithful to b.*

Proof We prove by induction on the complexity of A that v validates every node A, V' on b. For the case of literals, this is immediate from the definition of $v_{b,max}$ and $v_{b,min}$ and the readily verifiable fact that $C_b(\phi)$ always includes every number between its maximum and its minimum element. So suppose for induction that (IH) the claim holds for A, B, $\neg A$, and $\neg B$. It then suffices to show that it also holds for $(A \wedge B)$, $(A \vee B)$, $\neg(A \wedge B)$, $\neg(A \vee B)$, and $\neg\neg A$. Let Φ be any of these, and suppose that Φ, V' is on b. Since b is complete, the rule for Φ, V' has been applied. By inspection of the rules, every new node on b produced in applying that rule is occupied by A, B, $\neg A$, or $\neg B$, so by IH, v validates every such node. By lemma 23, v also validates Φ, V'. □

Lemma 26 *Let b be any complete 4-open branch of a web-tableau. Then*

1. *if b is i-open, then $v_{b,max}$ is complete,*
2. *if b is p-open, then $v_{b,min}$ is consistent, and*
3. *if b is 2-open, then some valuation v with $v_{b,min} \leq v \leq v_{b,max}$ is classical.*

Proof (1): Suppose $v^+_{b,max}(p) = 0$. Then by definition, $C_b(p) = \bigcap\{V' \subset V : (\phi, V')$ is on $b\} = \{0\}$. Since the only flags in a web-tableau are 3, < 2, and 0, it follows that $p, 0$ is on b. Since b is i-open, $\neg p, 0$ is not on b. So $0 \notin C_b(\neg p)$, and hence $v^-(p) = max(C_b(\neg p)) \neq 0$, as required.

(2): Suppose $v^+_{b,min}(p) \neq 0$. Then by definition, $0 \notin C_b(p) = \bigcap\{V' \subset V : (\phi, V')$ is on $b\}$. Since 0 belongs to every flag except 3, it follows that $p, 3$ is on b. Since b is p-open, $\neg p, 3$ is not on b, so $v^-_{b,min}(p) = min(C_b(\neg p)) = 0$.

(3): By part (1), $v_{b,max}$ is complete, and by part (2), $v_{b,min}$ is consistent. We let $v(p) = v_{b,max}(p)$ if that is classical. Otherwise, we let $v(p) = v_{b,min}(p)$ if that is classical. Finally, if $v_{b,max}(p)$ and $v_{b,min}(p)$ are both non-classical, $v_{b,max}(p)$ is inconsistent and $v_{b,min}(p)$ is incomplete, so we let $v(p) = (v^+_{b,max}(p), v^+_{b,min}(p))$, which is then classical. Then v is classical and $v_{b,min} \leq v \leq v_{b,max}$, as required. □

Theorem 27 *Let $\Gamma \cup \{C\}$ be a finite subset of L. Then $\Gamma \models_* C$ implies $\Gamma \vdash_* C$ for all $* \in \{4, I, P, 2\}$*

Proof Suppose that $\Gamma \nvdash_4 C$. Then there is a 4-open and complete web-tableau testing (Γ, C). Let b be a complete 4-open branch of that tableau. Note that b contains the node $C, < 2$ as well as $A, 3$ for each $A \in \Gamma$. Let v be any valuation with $v_{b,min} \leq v \leq v_{b,max}$. By lemma 25, v is faithful to b, so $v^+(A) = 3$ for all $A \in \Gamma$ and $v^+(C) < 2$. So $\Gamma \not\models_4 C$. The other cases are similar, using lemma 26 to obtain suitable—i.e. complete, consistent, or classical—valuations v with $v_{b,min} \leq v \leq v_{b,max}$ witnessing that $\Gamma \not\models_{I/C/P} C$. □

7 Conclusion

The aim of the present paper was to study and explore the notion of web consequence introduced in **?**. Schnieder's goal was to develop an explication of consequence which fits our pre-theoretical understanding of that notion better than the classical, modal explication, specifically by invalidating the principles that everything follows from a logical falsehood, and that a logical truth following from everything. His basic idea was to invoke the notion of *ground* for that purpose, taking a conclusion to be a consequence of some premises, roughly speaking, just when the premises, if true, provide a ground for the conclusion.

We saw that for the purposes of studying the logic of this consequence relation, a helpful simplification of this definition is obtained by working with a notion of ground that corresponds directly to the notion of a truthmaker within truthmaker semantics. So reframed, the proposal is to take P to be a logical consequence of Γ just in case in every scenario containing a truthmaker of Γ, some truthmaker of Γ—or equivalently, the maximal truthmaker of Γ—contains as a part a truthmaker of P. Depending on whether scenarios are assumed to be complete, consistent, both, or neither, we then obtain different variations of web consequence. The distinctive feature common to all of them is given by the general form of the definition, which we contrasted with a similar, but somewhat simpler form, on which P is taken to be a logical consequence of Γ iff every scenario containing a truthmaker of Γ contains a truthmaker of P. The difference in the structure of the definition was seen to lead to extensional differences in the consequence relation just in case scenarios are assumed to be complete. Generalizing over arbitrary, consistent, complete, or classical scenarios, the simpler definitions characterize the logics of FDE, K_3, LP, and classical logic, respectively, whereas the web consequence definitions yield FDE, K_3, Schnieder's preferred $\models_I$, and $\models_2$. Of these final two 'new' consequence relations, $\models_I$ seems the most interesting one, since it—like FDE—invalidates both of the abovementioned principles, but at the same time recognizes strictly more entailments than FDE.

To see whether $\models_I$ does an overall better job at matching a pre-theoretical understanding of consequence, we will need to look at principles on which the two logics differ, and which are simple and intuitively accessible enough that we may be said to have some pre-theoretical intuitions regarding their plausibility.[16] Here, I will merely highlight three central points over which the logics come apart, which may have some intuitive as well as theoretical significance. The first is a principle that, as we already mentioned, holds for $\models_I$ but not for FDE: that $A \vee (B \vee \neg B)$ entails $B \vee \neg B$. The other two are meta-rules that hold for FDE but not for $\models_I$. The first example is *Cut*: from the assumptions that Γ entails B, and Δ, B entails C, we may infer that Γ, Δ entails C. The second example is a standard rule of disjunction elimination: from the assumption that $\Gamma, P \vee Q$ entails R, we may infer that each of Γ, P and Γ, Q entails R. For $\models_I$, in contrast, we only have a version of this rule that is modified in a way analogous to the tableaux rule for disjunction: from the assumption that $\Gamma, P \vee Q$ entails R, we may infer that each of Γ, P, Q and $\Gamma, P, \neg Q$ and $\Gamma, \neg P, Q$ entails R.[17]

My own inclination is to take all three points to speak in favour of FDE, but this is not the place to argue the point.

[16] Needless to say, fit with pre-theoretical intuition is by no means the only criterion by which to evaluate a proposed account of consequence.

[17] I am not aware of a natural weakening of Cut that holds for $\models_I$.