TRUTHMAKER EQUIVALENCE

[BLINDED]

Abstract. Truthmaker semantics provides a powerful framework for the study of hyperintensional operators: operators that distinguish between some pairs of intensionally equivalent contents. Even under a truthmaker based account though, an operator will not distinguish between *truthmaker equivalent* contents. This paper determines the propositional logics of truthmaker equivalence for a range of related conceptions of truthmaking, supplementing previous results on this topic obtained by Kit Fine and Fabrice Correia.

1. Introduction

Truthmaker semantics is a powerful formal framework for the study of *hyperintensional* operators.[1] To give just some examples, it has been used with considerable success to develop hyperintensional accounts of the counterfactual conditional (Fine, 2012a), the notion of partial content (Fine, 2016, 2017a), the notion of ground (Fine, 2012c,b, 2017b), imperatives (Fine, 2018a,b), the notions of permission and obligation (Fine, 2018b; Anglberger et al., 2016), the notion of evidential relevance (Krämer, 2017), and the notion of a whole truth (Krämer, 202x). The distinctive feature of hyperintensional operators is that necessarily equivalent sentences are not in general substitutable *salva veritate* within their argument places. Thus, suppose $\varphi(\ldots)$ is a one-place, sentence-forming operator on sentences. Then if φ is hyperintensional, even if A and B are necessarily equivalent sentences, $\varphi(A)$ may be true and yet $\varphi(B)$ false, or the other way around. If so, the semantics of φ cannot be captured within the intensional framework of standard possible world semantics. In that framework, necessarily equivalent sentences are assigned the same semantic content – viz., the same set of possible worlds – and so there is no way for the semantics to distinguish between $\varphi(A)$ and $\varphi(B)$, and thus for them to receive different truth-values.

Truthmaker semantics provides a more fine-grained account of the content of sentences than possible world semantics. As a result, within this framework, necessarily and even logically equivalent sentences are often assigned different contents. So the mere necessary equivalence of A and B no longer prevents us from distinguishing

[1] [blinded]

within our semantics between $\varphi(A)$ and $\varphi(B)$, or from assigning different truth-values to them. But although necessary equivalence does not imply semantic indistinguishability within truthmaker semantics, there is a different (non-trivial[2]) equivalence relation which does – call it *truthmaker equivalence*: If two sentences A and B are truthmaker equivalent, they are assigned the same content even within truthmaker semantics, and any truthmaker based account of φ will have the consequence that $\varphi(A)$ and $\varphi(B)$ must have the same truth-value. In order to identify the predictions of a proposed truthmaker semantics for an operator φ, it is therefore important to know when two sentences are truthmaker equivalent.

Generally speaking, the considerations relevant to answering this question may vary with the subject matter of the specific sentences we are looking at. For instance, in order to say whether '2+2=4' and '8/2=4' are truthmaker equivalent, we may need to determine what the truthmakers of arithmetical truths are, and how they are individuated. But in some cases, general considerations about the *logical* structure of a pair of sentences suffice to show that the sentences are truthmaker equivalent. The focus of this paper will be on the question when the *propositional logical form* of two sentences guarantees their truthmaker equivalence.[3]

As it turns out, there are a number of subtly different interpretations we can give to the notion of truthmaker equivalence. Which of them is most relevant, and most tightly connected to the matter of intersubstitutability, depends on the specific application of truthmaker semantics, i.e. on the specific account of a given hyperintensional operator that is under consideration. For some of these notions, the logics of equivalence have already been determined, but not for some others.[4] In particular, the logic for what is arguably the most central, and most discriminating notion of truthmaker equivalence has not been established in the existing literature.[5] My aim in this paper is, firstly, to close this gap, and secondly, to offer a single, comprehensive point of reference on the

[2] Identity is an equivalence relation which will trivially ensure interchangeability *salva veritate* of equivalent sentences. But truthmaker equivalence, as we shall see, is weaker than identity.

[3] While proposals for a truthmaker semantics for first-order logic exist, the question of how best to deal with quantifiers within truthmaker semantics is considerably harder and more contentious than the treatment of the truth-functional operators. More importantly, the question of the logic of the truthmaker equivalence under the existing proposals is very difficult.

[4] The relevant papers are Fine (2016) and Correia (2016). I will be more specific about what is established in these paper in due course.

[5] (Fine, 2018a: p. 625) claims of a certain deductive system that it is sound and complete for this notion of equivalence, but as I show below (in footnote 20), this is a mistake, as the system in question is actually not complete.

logic of truthmaker equivalence, describing the logics for all the different conceptions of truthmaker equivalence and discussing their interrelations.

The plan for the paper is as follows. After introducing the framework of truthmaker semantics (2), I explain the relevant different conceptions of truthmaking and truthmaker equivalence, and describe applications of truthmaker semantics for which they may plausibly be regarded as pertinent (§3). I then specify, for each of these different conceptions, a deductive system for deriving the logically true equivalences between sentences of a propositional language and prove soundness and independence results (§4), before establishing completeness (§§5-7).

2. Truthmaker Semantics

The best way to introduce truthmaker semantics is to describe how it deviates from the familiar framework of possible world semantics.[6] Firstly, instead of a space of all the possible worlds, we work with a space of *states*. Like worlds, states are *specific*, non-disjunctive entities. Unlike worlds, they need not be *complete*, so they need not decide the truth-value of every proposition. They also need not be *possible*, so they can make true contradictions and other necessary falsehoods. Secondly, while the notion of a possible world's making true a proposition may be understood in modal terms – that the world *necessitates* the truth of the proposition – we will work with a notion of a state's *exactly making true*, or *exactly verifying* a proposition that is much narrower than necessitation. Exact verification is subject to the requirement that the state be *relevant as a whole* to the proposition in question. Thus, the state of grass being green does not verify the proposition that snow is white or not, because it is wholly irrelevant to the truth of that proposition. And while the state of it being sunny exactly verifies the proposition that it is sunny, the state of it being sunny and hot does not, because it is not relevant *as a whole*: it contains as an irrelevant part the state of it being hot. So we can see that exact verification is non-monotonic: a state may contain an exact verifier without itself being an exact verifier. In addition, the sense in which the obtaining of a state is required to be *sufficient* for the truth of a proposition it exactly verifies is more demanding than necessitation, and akin to the characteristic way in which the truth of *full grounds* are sufficient for the truth of what they ground. For instance, the state of Socrates being wise does not exactly verify that Socrates is wise and snow is white or not, even though it is both wholly relevant and such that it necessitates the truth of the proposition: it lacks a part that is sufficient in the right way for snow's being white or not.

[6] For a more in-depth presentation, the best source is Fine (2017a).

As can be inferred from the previous remarks, states are taken to enter into *part-whole* relations. We write $s \sqsubseteq t$ to say that state s is part of state t. We assume that for any set of states $T = \{t_1, t_2, \ldots\}$ there exists their *fusion* $\bigsqcup T = t_1 \sqcup t_2 \sqcup \ldots$, which we take to be the smallest state to contain all of $t_1, t_2, \ldots$ as part. More precisely, the basic structure in truthmaker semantics is that of a *state-space*, which is defined as follows.

Definition 1. *A* state-space *is a pair* $(S, \sqsubseteq)$ *such that*

(1) S *is a non-empty set,*

(2) $\sqsubseteq$ *is a partial order*[7] *on* S *such that for any subset* T *of* S*, there exists the least upper bound*[8] $\bigsqcup T$ *of* T *with respect to* $\sqsubseteq$

Relative to a state-space, we then define the notion of a proposition.

Definition 2. *A* unilateral proposition *on a state-space* $(S, \sqsubseteq)$ *is any subset of* S*.*

Under this definition, a proposition is thus identified with the set if its verifiers. Note that the definition reflects the conception of exact verification as non-monotonic: there is no requirement that if P is a proposition and $s \in P$, then any state t with $s \sqsubseteq t$ should also be a member of P.

One might question, however, whether some weaker closure principles should not be imposed. In particular, one might consider the following closure constraints (cf. e.g. (Fine, 2017a: p. 628f)):[9]

Convexity: t verifies P whenever $s \sqsubseteq t \sqsubseteq u$ and s and u verify P

Closure under Fusion: $s \sqcup t \sqcup \ldots$ verifies P whenever all of $s, t, \ldots$ verify P

For reasons I shall give below, I think that exact verification does not in general satisfy either of these constraints. At the same time, as we shall see, it is sometimes useful to focus not on exact verification itself, but one derivative, more inclusive notions of verification which by definition satisfy these constraints.

Definition 3. *For* s *a state and* P *a unilateral proposition, say that*

- s convexity-inclusively *(short: c-inclusively) verifies* P *iff* $t \sqsubseteq s \sqsubseteq u$ *for some states* t *and* u *verifying* P

7 Recall that $\sqsubseteq$ is a partial order on S iff $\sqsubseteq$ is a binary relation on S that is reflexive ($s \sqsubseteq s$ for all $s \in S$), transitive ($s \sqsubseteq u$ whenever $s \sqsubseteq t$ and $t \sqsubseteq u$) and anti-symmetric ($s = t$ whenever $s \sqsubseteq t$ and $t \sqsubseteq s$).

8 A state s is an *upper bound* of $T \subseteq S$ iff $t \sqsubseteq s$ for all $t \in T$, and s is a least upper bound of T iff s is an upper bound of T and $s \sqsubseteq u$ whenever u is an upper bound of T. Least upper bounds are easily seen to be unique if they exist.

9 Another constraint Fine discusses that one might wish to impose is that of *verifiability* (ibid.), so that only non-empty sets of states are taken to represent propositions. For our purposes, it makes no difference whether the constraint is imposed.

- s fusion-inclusively *(short: f-inclusively) verifies P iff s is the fusion of some verifiers of P*
- s regularly *verifies P iff* $t \sqsubseteq s \sqsubseteq \bigsqcup P$ *for some state t verifying P.*

Thus, the c-inclusive verifiers are exactly the states that lie *between* some verifiers, the f-inclusive verifiers are the states that may be obtained by *fusing* verifiers, and the regular verifiers are the states that lie between a verifier and the fusion of all verifiers. Note that both every f-inclusive verifier and every c-inclusive verifier is a regular verifier, so regular verification is the least discriminating among the four conceptions of verification. However, some c-inclusive verifiers are not f-inclusive verifiers and some f-inclusive verifiers are not c-inclusive verifier. For let p, q, r be pairwise distinct and consider the state-space $(\wp(\{p, q, r\}), \subseteq)$. Then $\{p, q\}$ is a c- but not f-inclusive verifier of $\{\{p\}, \{p, q, r\}\}$, and $\{p, q\}$ is an f- but not c-inclusive verifier of $\{\{p\}, \{q\}\}$. So in terms of fineness of grain of the four conceptions of verification, we have the following structure

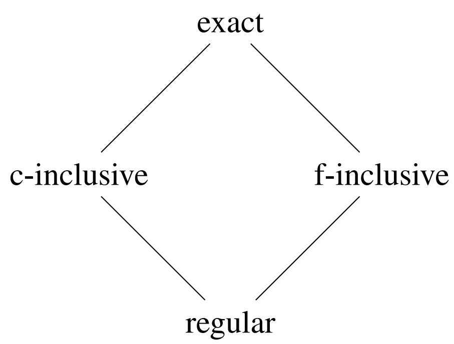

Figure 1: *Varieties of Verification*

with maximal fineness of grain at the top node, corresponding to exact verification.[10]

Given the view of propositions as sets of their exact verifiers, we may define define operations of conjunction and disjunction as follows:

Definition 4. *For P, Q unilateral propositions on some state-space,*

- $P \wedge Q = \{s \sqcup t : s \in P \text{ and } t \in Q\}$
- $P \vee Q = P \cup Q$

[10] Note that the term 'exact verification' is sometimes applied to any form of non-monotonic verification. Correspondingly, Fine calls a state an *inexact* verifier of a proposition iff it contains an exact verifier (cf. e.g. 2017c: p. 558). So I should emphasize that although none of c-inclusive, f-inclusive, and regular verification are inexact forms of verification, in my terminology they are distinct from what I call exact verification.

Note how the definition of disjunction reflects the conception of states as non-disjunctive: the only way for a state to get to verify a disjunction is by verifying one of its disjuncts. Again, this is just as for worlds within possible worlds semantics. The definition of conjunction in turn reflects the requirement that a verifier be *wholly relevant* to what it verifies. For note that in contrast to its possible worlds counterpart, the clause is not that a state verifies a conjunction iff it verifies both conjuncts. The reason is that even for conjuncts that are perfectly compatible with one another – like the propositions that snow is white and that grass is green, say – there will not in general be a state that is both sufficient for and wholly relevant to both conjuncts. So instead, a state is counted as a verifier of a conjunction if it may be obtained by fusing a verifier of one conjunct with a verifier of the other.

Negation is somewhat trickier to handle than conjunction and disjunction. Since states are not subject to a completeness requirement, any proposition divides the states into three typically non-empty sets: those that verify it, those that verify its negation, and those that do neither. As a result, we cannot in any straightforward way read off from the set of verifiers of a proposition the set of states verifying its negation.

We may deal with this difficulty by adopting a *bilateral* conception of propositions, specifying separately both the set of verifiers and the set of falsifiers, and taking the verifiers (falsifiers) of the negation of a proposition P to be the falsifiers (verifiers) of P. For the falsifiers of conjunctions and disjunctions, we adopt the obvious 'dual' clauses, so that a state falsifies a conjunction iff it falsifies either conjunct, and it falsifies a disjunction iff it is the fusion of a falsifier of one disjunct with a falsifier of the other disjunct.

Definition 5. *A* bilateral proposition $\mathbf{P}$ *on a state-space* $(S, \sqsubseteq)$ *is any pair of unilateral propositions on* $(S, \sqsubseteq)$*. If* $\mathbf{P}$ *is a bilateral proposition,* $\mathbf{P}^+$ *(*$\mathbf{P}^-$*) is its first (second) coordinate, and will be called the positive (negative) component or content of* $\mathbf{P}$.

Definition 6. *For* $\mathbf{P}, \mathbf{Q}$ *bilateral propositions on some state-space,*

- $\mathbf{P} \wedge \mathbf{Q} = (\mathbf{P}^+ \wedge \mathbf{Q}^+, \mathbf{P}^- \vee \mathbf{Q}^-)$
- $\mathbf{P} \vee \mathbf{Q} = (\mathbf{P}^+ \vee \mathbf{Q}^+, \mathbf{P}^- \wedge \mathbf{Q}^-)$
- $\neg \mathbf{P} = (\mathbf{P}^-, \mathbf{P}^+)$

This completes my exposition of the basic framework of truthmaker semantics. It is easy to see that, due to the move from possible worlds to (possible or impossible) states, and from necessitation to exact verification, the framework allows us to make much more fine-grained distinctions between propositions than are available within the standard framework of possible worlds semantics. For example, the logically true proposition that snow is white or not is no longer identified with the logically true proposition that grass is green or not: the state of snow's being white verifies the former but not the latter. The contradictory proposition that snow is both white and not white is no longer identified with the logically false proposition that grass is both green and not green: the fusion of the state of snow's being white and the state of snow's being red verifies the former but not the latter.

Note that these two examples do not even depend on the non-monotonicity of exact verification. For an example that does, consider the proposition that snow is white and the proposition that snow is white or (snow is white and grass is green). They are distinguished within truthmaker semantics because the latter, but not the former is verified by the state that snow is white and grass is green. But since the former is verified by the state that snow is white, its failure to be verified by the state that snow is white and grass is green is an instance of exact verification's non-monotonicity. The example is of some significance because it illustrates how truthmaker equivalence is more demanding than equivalence in familiar relevant logics, such as the logic of first-degree entailment, for these do render pairs of the form A and $A \vee (A \wedge B)$ equivalent.[11]

3. Varieties of Truthmaker Equivalence

Within the general framework of truthmaker semantics just outlined, we may distinguish a variety of equivalence relations between propositions. The strongest, most demanding one is simply numerical identity, requiring sameness both of verifiers and falsifiers of a bilateral proposition. Then there are two ways in which this requirement may be weakened. The first is to disregard one of the two components in our bilateral propositions, and require only sameness of verifiers, or only sameness of falsifiers. We may call equivalence relations considering both components, only the positive component, or only the negative component of a bilateral proposition *full*, *positive*, and *negative* equivalence relations, respectively.

These relations differ in how they bear on the question of intersubstitutability. Suppose we are considering some propositional language under a bilateral truthmaker semantics. Obviously, then, any two sentences A and B that are fully equivalent will be freely interchangeable. Not only will the truth-value of $C(A)$ and $C(B)$ be the same for any context C, but $C(A)$ and $C(B)$ themselves will also have the same verifiers and falsifiers, i.e. be fully equivalent.

[11] Indeed, it is known that the logic of *inexact truthmaker* equivalence is the logic of mutual first-degree entailment, where a state inexactly verifies a proposition iff it contains an exact verifier of the proposition; cf. (Fine, 2016: sec. 10).

However, a certain degree of interchangeability may be guaranteed already by positive or negative equivalence. For instance, in many cases, the semantics for an operator φ will be such that the positive content of $\varphi(A)$ depends only on the positive content of A. This holds, for example, for conjunction and disjunction, but also for many other operators for which truthmaker semantical accounts have been proposed.[12] In such a case, the positive equivalence of A and B will guarantee sameness of positive content of $\varphi(A)$ and $\varphi(B)$. And since the truth-value of a sentence will typically be fixed by the positive content – a sentence being true iff one of its truthmakers obtains – the positive equivalence of two sentences will often be enough to tell us that a certain substitution of one for the other is bound to preserve truth-value.

It is therefore of considerable interest to know when two sentences are bound by their logical form to be positively equivalent, to have the same truthmakers, even if their falsitymakers differ. Fortunately, once we have found the answer to this question, we can easily extend it to also answer the corresponding questions for negative and full equivalence. For A and B are bound by their logical form to be negatively equivalent iff $\neg A$ and $\neg B$ are bound by their logical form to be positively equivalent, and they are bound to be fully equivalent iff bound to be both positively and negatively equivalent. In what follows, we shall therefore focus on positive rather than negative or full equivalence.

The second way to weaken the requirements for equivalence is to appeal to the more coarse-grained conceptions of c-inclusive, f-inclusive, or regular truthmaking introduced before, and define corresponding relations of equivalence as sameness of c-inclusive, f-inclusive, or regular verifiers and/or falsifiers. For any unilateral proposition P, let P^C, P^F, and P^R denote the set of c-inclusive, f-inclusive, and regular verifiers of P, respectively.[13]

Definition 7. *For P, Q unilateral propositions, say that*

- *P is* exactly equivalent *to Q ($P \approx_E Q$) iff $P = Q$*
- *P is* c-equivalent *to Q ($P \approx_C Q$) iff $P^C = Q^C$*
- *P is* f-equivalent *to Q ($P \approx_F Q$) iff $P^F = Q^F$*

---

12 This is not to say, of course, that operators that do not have this feature would have to be peculiar in some way. Most obviously, negation does not have this feature, since it flips verifiers and falsifiers.

13 It is worth noting that P^C, P^F, and P^R have an equivalent characterization in terms of corresponding closure properties on propositions. Thus, say that P is *convex* iff $t \in P$ whenever $s \sqsubseteq t \sqsubseteq u$ and $s, u \in P$, *closed under fusion* iff $\bigsqcup T \in P$ whenever $\emptyset \subset T \subseteq P$, and regular iff convex and closed under fusion. It is routine to verify that P^C, P^F, and P^R are the smallest supersets of P that are convex, closed under fusion, and regular, respectively. We may also note that $P^R = (P^F)^C$, though not in general $P^R = (P^C)^F$. For example, if s and $t \sqcup u$ are states neither of which is part of the other, then $(\{s, t \sqcup u\}^C)^F = \{s, t \sqcup u\}^F = \{s, t \sqcup u, s \sqcup t \sqcup u\} \neq \{s, t \sqcup u, s \sqcup t, s \sqcup u, s \sqcup t \sqcup u\} = \{s, t \sqcup u\}^R$.

- *P is* r-equivalent *to Q ($P \approx_R Q$) iff $P^R = Q^R$*

Which of these relations is most important, and most intimately connected to the question of intersubstitutability, depends on the application of truthmaker semantics under consideration, and whether it depends on our ability to distinguish between r-equivalent, c-equivalent, and/or f-equivalent propositions. For example, it is plausible that for the purposes of a theory of *partial content*, we need not distinguish between r-equivalent propositions. Fine (2016, 2017a: sec. 5) has defended an explication of the notion of one (unilateral[14]) proposition being part of another by the condition that (i) every verifier of the first is part of a verifier of the second, and (ii) every verifier of the second has a verifier of the first as a part. It can be shown that r-equivalent propositions enter into the same relations of parthood, so understood. Relatedly, parthood is better behaved as a relation on the set of regular propositions rather than an on the set of all propositions, since it is guaranteed to be anti-symmetric only on the set of regular propositions. So for the study of partial content, it is appropriate to work with the concept of regular verification, and the most important concept of equivalence is that of r-equivalence.[15] Similar points apply to the theory of partial truth in Fine (202x) and the theory of verisimilitude in Fine (2019).

But for other applications, we need to work with a more fine-grained conception of verification, and thus of equivalence. One such application is the theory of *metaphysical grounding*. As Fine has shown (2012c; 2012b; 2017b), we can give a plausible explication of a worldly notion of ground within the truthmaker framework. Briefly, Fine proposes to take some propositions $P_1, P_2, \ldots$ to *weakly, non-factively* ground a proposition Q iff $s_1 \sqcup s_2 \sqcup \ldots$ verifies Q whenever $s_1, s_2, \ldots$ verify $P_1, P_2, \ldots$, respectively. We then take $P_1, P_2, \ldots$ to *strictly, non-factively* ground Q iff there are no propositions together with which Q weakly grounds any of $P_1, P_2, \ldots$. The ordinary factive notion of ground is then obtained by demanding that all the P_i – and hence Q – be true.

Now we obtain subtly different accounts of ground depending on whether verification is here understood as exact, f-inclusive, c-inclusive, or regular. It has been argued, independently but for the same reasons, both in Krämer and Roski (2015) and Correia (2016), that adopting the regular or c-inclusive conception of truthmaking leads to implausible results. To see why, consider the truth X that Frege is an American philosopher or Quine is an American philosopher. If verification is at least f-inclusive, the claim that

14 For bilateral propositions, it may also be required that every falsifier of the second proposition is also a falsifier of the first; cf. (Fine, 2017a: p. 643).

15 It is for this reason that Fine (2016) focuses on the logic of what I have called r-equivalence; cf. (Fine, 2016: sec. 4).

Frege is an American philosopher and Quine is an American philosopher comes out a non-factive ground of X, prevented from being a ground only by its falsity. If verification is also c-inclusive, however, the claim that Frege is a philosopher and Quine is an American philosopher – since it 'lies between' the claim that Quine is an American philosopher and the claim that Frege and Quine are both American philosophers – is also a non-factive ground. And since this claim is true, it comes out an actual ground of X, and so it follows that the truth that Frege is a philosopher helps ground the truth that either Frege or Quine is an American philosopher. But this seems implausible. For it seems that Frege's being a philosopher could only play a role in bringing it about that Frege or Quine is an American philosopher in conjunction with Frege's being American, which he is not. Since this point applies also with respect to the truth that Frege is an American philosopher, or Quine is an American philosopher, or Frege and Quine are both American philosophers, dropping f-inclusiveness will not help, and so we need to drop c-inclusiveness, and work with either the exact or the f-inclusive conception of verification.

The logics of regular and f-equivalence are known; they are described in Fine (2016) and Correia (2016), respectively. But there are also applications for which even f-inclusiveness needs to be dropped. One plausible example is the application to what may be called the *logic of totality*. Krämer (202x) argues that truthmaker semantics allows us to give a more satisfactory account of the so-called totality operator '... and that's it' than is available within intensional frameworks. As he points out, it seems that this operator is sensitive to the exact truthmakers of its argument, and not just its f-inclusive ones. For there is a natural reading of the totality operator under which the statements

(1) I had a dram of Lagavulin or a dram of Ardbeg, and that's it.
(2) I had a dram of Lagavulin, or a dram of Ardbeg, *or a dram of each*, and that's it.

have different truth-conditions, with only the latter being compatible with me having had both a dram of Lagavulin and a dram of Ardbeg. But in order to assign such truth-conditions, we need to allow that the embedded sentence in (1) is verified by the state of me having a dram of Lagavulin, and by the state of me having a dram of Ardbeg, but not by the fusion of these two states.

Similarly, there is a natural reading of

(3) I had coffee, or coffee and eggs and bacon, and that's it.
(4) I had coffee, or coffee and eggs, and coffee and eggs and bacon, and that's it.

under which they have different truth-conditions, with only the latter allowing for a scenario in which I had eggs but no bacon. But in order to assign such truth-conditions, we need to allow that the embedded sentence in (3) is verified by the state c of me having coffee, and by the state $c \sqcup e \sqcup b$ of me having coffee, eggs, and bacon, but not by the state $c \sqcup e$ in between those two. So to capture these readings of the totality operator, we need to work with the exact notion of truthmaking, assuming neither c- nor f-inclusiveness.

Similar examples support the claim that truthmaker based theories of imperatives and of deontic operators of permission and obligation need to distinguish between f- and/or c-equivalent propositions (cf. Fine (2018a,b)). The permission to have ice-cream or chocolate is not automatically a permission to have both. By shutting the door and closing the window, one may fail to comply with the imperative to shut the door or close the window. As a result, as Fine points out (2018a: sec. 6), the propositional logic of exact equivalence constitutes an essential component of a fully developed truthmaker account of these notions.

So in addition to regular and f-equivalence, we have strong reasons to be interested in the logic of exact equivalence as well. I am not aware of a compelling example of an operator that does not distinguish between c-equivalent propositions but that does distinguish between r-equivalent propositions. From a technical point of view, however, it is a very natural further question in the present context what the logic of c-equivalence is, and how it relates to the others.[16]

For exact truthmakers, it is easily seen that since disjunction is defined in terms of set-theoretic union, we can express the subset-relation between propositions in terms of disjunction and exact truthmaker equivalence: $P \subseteq Q$ iff $P \vee Q \approx_E Q$. We shall later have occasion to make use of the fact that this relationship extends to f-inclusive, c-inclusive, and regular truthmakers:

Proposition 1. *For P, Q unilateral propositions:*

(1) $P^C \subseteq Q^C$ *iff* $P \vee Q \approx_C Q$
(2) $P^F \subseteq Q^F$ *iff* $P \vee Q \approx_F Q$
(3) $P^R \subseteq Q^R$ *iff* $P \vee Q \approx_R Q$

Proof. For (1), left-to-right: Suppose $P^C \subseteq Q^C$. We need to show that $(P \cup Q)^C = Q^C$. Suppose first that $t \in (P \cup Q)^C$. Let $s \sqsubseteq t \sqsubseteq u$ with $s, u \in P \cup Q$. Then both s and u are either in P or in Q. But $Q \subseteq Q^C$ and $P \subseteq P^C \subseteq Q^C$ by assumption, so s and u are both in Q^C, and hence so is t. Suppose now that $t \in Q^C$. Then there are $s, u \in Q \subseteq P \cup Q$, with $s \sqsubseteq t \sqsubseteq u$, so $t \in (P \cup Q)^C$ as desired. So $(P \cup Q)^C = Q^C$.

[16] In particular, given that a set of states is regular iff both closed under fusion and convex, it is natural to ask whether the logic of r-equivalence can be obtained by combining the logics of f- and c-equivalence. As we shall prove below, this is indeed the case.

Right-to-left: Suppose that $(P \cup Q)^C = Q^C$. We wish to show that $P^C \subseteq Q^C$. Suppose $t \in P^C$, so $s \sqsubseteq t \sqsubseteq u$ for some $s, u \in P$. Then $s, u \in P \cup Q$, hence $t \in (P \cup Q)^C$. By assumption, it follows that $t \in Q^C$.

For (2), left-to-right: Suppose $P^F \subseteq Q^F$. We need to show that $(P \cup Q)^F = Q^F$. Suppose first that $s \in (P \cup Q)^F$. Let $s = \bigsqcup T$ with $T \subseteq P \cup Q$. Let $T_P = T \cap P$ and $T_Q = T \cap Q$. Then $s = \bigsqcup T_P \sqcup \bigsqcup T_Q$. But $\bigsqcup T_P \in P^F$, hence $\bigsqcup T_P \in Q^F$. So for some $W \subseteq Q$, $\bigsqcup T_P = \bigsqcup W$. But then $s = \bigsqcup W \sqcup \bigsqcup T_Q = \bigsqcup (W \cup T_Q)$. And since $W \cup T_Q \subseteq Q$, it follows that $s \in Q^F$. Now suppose $s \in Q^F$. Let $s = \bigsqcup T$ with $T \subseteq Q$. Then $T \subseteq P \cup Q$, hence $s = \bigsqcup T \in (P \cup Q)^F$.

Right-to-left: Suppose $(P \cup Q)^F = Q^F$ and suppose $s \in P^F$. Let $s = \bigsqcup T$ with $T \subseteq P$. Then $T \subseteq P \cup Q$, so $s = \bigsqcup T \in (P \cup Q)^F = Q^F$, as desired.

For (3), left-to-right: Suppose $P^R \subseteq Q^R$. We wish to show that $(P \cup Q)^R = Q^R$. Suppose first that $t \in (P \cup Q)^R$. Let $s \sqsubseteq t \sqsubseteq \bigsqcup (P \cup Q)$ with $s \in P \cup Q$. Then by assumption, $s \in Q^R$. Moreover, by assumption, $\bigsqcup P \sqsubseteq \bigsqcup P^R \sqsubseteq \bigsqcup Q^R$. And since also $\bigsqcup Q \sqsubseteq \bigsqcup Q^R$, $\bigsqcup (P \cup Q) = \bigsqcup P \sqcup \bigsqcup Q \sqsubseteq \bigsqcup Q^R$. So $s \sqsubseteq t \sqsubseteq \bigsqcup Q^R$ and $s \in Q^R$, hence by regularity of Q^R, $t \in Q^R$. Suppose now that $t \in Q^R$ and let $s \sqsubseteq t \sqsubseteq \bigsqcup Q$ with $s \in Q$. So $s \in P \cup Q$, and since $\bigsqcup Q \sqsubseteq \bigsqcup (P \cup Q)$, it follows that $t \in (P \cup Q)^R$.

Right-to-left: Suppose $(P \cup Q)^R = Q^R$. We wish to show that $P^R \subseteq Q^R$. Suppose $t \in P^R$, and let $s \sqsubseteq t \sqsubseteq \bigsqcup P$ with $s \in P$. Then $s \in (P \cup Q)^R$ and thus by assumption $s \in Q^R$ as well as $\bigsqcup P \sqsubseteq \bigsqcup (P \cup Q)$ and $\bigsqcup (P \cup Q) \in Q^R$, so by regularity of Q^R, it follows that $t \in Q^R$. □

4. Deductive Systems, Soundness, and Independence

Let $\mathcal{L}$ be a standard propositional language with connectives $\wedge$, $\vee$, $\neg$. We call the formulas of $\mathcal{L}$ the *$\mathcal{L}$-sentences*, and call an *$\mathcal{L}$-equivalence* any expression of the form $A \approx B$ with A, B $\mathcal{L}$-sentences. The *language of equivalence* $\mathcal{L}_\approx$ over $\mathcal{L}$ is the set of all $\mathcal{L}$-equivalences.

Definition 8. *An $\mathcal{L}$-model is any triple $\mathcal{M} = (S, \sqsubseteq, [\cdot])$ such that $(S, \sqsubseteq)$ is a state-space, and $[\cdot]$ maps every atomic $\mathcal{L}$-sentence to a bilateral proposition on $(S, \sqsubseteq)$. Given an $\mathcal{L}$-model, $[\cdot]$ is extended to the complex formulas of $\mathcal{L}$ in the obvious way:*

- $[\neg A] = \neg[A]$
- $[A \wedge B] = [A] \wedge [B]$
- $[A \vee B] = [A] \vee [B]$

Relative to an $\mathcal{L}$-model, we then define the exact, c-inclusive, f-inclusive, and regular content of an $\mathcal{L}$-sentence A:[17]

Definition 9. *Given an $\mathcal{L}$-model $\mathcal{M} = (S, \sqsubseteq, [\cdot])$, for any $\mathcal{L}$-sentence A, the*

- exact *content $[A]_e$ of A is $[A]$*
- c-inclusive *content $[A]_c$ of A is $(([A]^+)^C, ([A]^-)^C))$*
- f-inclusive *content $[A]_f$ of A is $(([A]^+)^F, ([A]^-)^F))$*
- regular *content $[A]_r$ of A is $(([A]^+)^R, ([A]^-)^R))$*

Definition 10. *Given an $\mathcal{L}$-model $\mathcal{M} = (S, \sqsubseteq, [\cdot])$, an $\mathcal{L}$-equivalence $A \approx B$ is* true *in $\mathcal{M}$ under the*

- exact *interpretation of $\approx$ iff $[A]^+ \approx_E [B]^+$*
- c-inclusive *interpretation of $\approx$ iff $[A]^+ \approx_C [B]^+$*
- f-inclusive *interpretation of $\approx$ iff $[A]^+ \approx_F [B]^+$*
- regular *interpretation of $\approx$ iff $[A]^+ \approx_R [B]^+$*

Definition 11. *An $\mathcal{L}$-equivalence $A \approx B$ is* valid *under an interpretation of $\approx$ iff for every $\mathcal{L}$-model $\mathcal{M}$, $A \approx B$ is true in $\mathcal{M}$ under that interpretation of $\approx$. We write $\models_e$ ($\models_c$, $\models_f$, $\models_r$) $A \approx B$ to say that $A \approx B$ is valid under the interpretation of $\approx$ as exact (c-inclusive, f-inclusive, regular) equivalence.*

Before moving on, we note that by using disjunction, we can express within $\mathcal{L}_\approx$ not only *identity* but also *subset*-relations between exact, c-inclusive, f-inclusive, or regular content. For it is immediate from proposition 1 above that

Proposition 2. *An $\mathcal{L}$-equivalence $A \vee B \approx B$ is true under*

- *the exact interpretation of $\approx$ iff $[A]_e^+ \subseteq [B]_e^+$*
- *the c-inclusive interpretation of $\approx$ iff $[A]_c^+ \subseteq [B]_c^+$*
- *the f-inclusive interpretation of $\approx$ iff $[A]_f^+ \subseteq [B]_f^+$*
- *the regular interpretation of $\approx$ iff $[A]_r^+ \subseteq [B]_r^+$*

4.1. **Deductive Systems.** We shall now specify deductive systems for the different readings of $\approx$.The *base system* $\mathfrak{D}_b$ consists of the following axioms and rules.

Collapse($\vee$)	$A \vee A \approx A$
Commutativity($\vee$)	$A \vee B \approx B \vee A$
Associativity($\vee$)	$A \vee (B \vee C) \approx (A \vee B) \vee C$
Commutativity($\wedge$)	$A \wedge B \approx B \wedge A$

[17] In (Fine, 2016: p. 208), the regular content of a formula is called its *replete* content, and the f-inclusive content is called the *complete* content; there is no special term for the c-inclusive content.

Associativity(∧)	$A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$
Collapse(¬¬)	$\neg\neg A \approx A$
DeMorgan(¬∨)	$\neg(A \vee B) \approx \neg A \wedge \neg B$
DeMorgan(¬∧)	$\neg(A \wedge B) \approx \neg A \vee \neg B$
Distributivity(∧/∨)	$A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$
Preservation(∨)	$A \approx B \ / \ A \vee C \approx B \vee C$
Preservation(∧)	$A \approx B \ / \ A \wedge C \approx B \wedge C$
Symmetry	$A \approx B \ / \ B \approx A$
Transitivity	$A \approx B,\ B \approx C \ / \ A \approx C$

For any equivalence $A \approx B$ in $\mathcal{L}_{\approx}$, we write $\vdash_b A \approx B$ to say that $A \approx B$ can be derived using only the rules and axioms of $\mathfrak{D}_b$, and similarly for the various extensions of $\mathfrak{D}_b$ to be considered below.[18]

The system $\mathfrak{D}_b$ coincides with the system used in (Fine, 2016: p. 201) minus Fine's axioms E2 and E11. Fine shows (ibid, p. 202) that a rule of Positive Replacement is an admissible rule in $\mathfrak{D}_b$, in the sense that it preserves theoremhood. We can state the rule as follows. If A, B, C are sentences of $\mathcal{L}$, let $C(A/B)$ be the result of replacing the occurrences of A in C by B. Then the rule of Positive Replacement is

PR $\quad A \approx B \ / \ C \approx C(A/B)$, if no occurrences of A in C are in the scope of ¬

It is easy to verify that if a sentence D may be obtained from C by replacing one or more, but not necessarily all occurrences of A in C by B, then as long as the replaced occurrences are not in the scope of ¬, $C \approx D$ will also be derivable from $A \approx B$, and I shall often tacitly rely on this fact in applying PR. As becomes clear on inspection of Fine's proof of the admissibility of PR, it depends only on the availability of the commutativity axioms and the preservation rules for ∧ and ∨. PR therefore remains admissible under the various extensions of $\mathfrak{D}_b$ to be considered in what follows.

We shall abbreviate formulas of $\mathcal{L}_{\approx}$ of the form $A \vee B \approx B$ as $A \rightarrow B$. By proposition 2, $A \rightarrow B$ will be true under a given interpretation of $\approx$ iff the content of A pertinent to that interpretation is a subset of the relevant content of B. The *exact* system $\mathfrak{D}_e$ is then obtained by adding to $\mathfrak{D}_b$ the *exact collapse* axiom for ∧:

ECollapse(∧) $\quad A \rightarrow A \wedge A$

In effect, this axiom registers the fact that since fusing a state with itself yields that same state, every truthmaker of A is also a truthmaker of $A \wedge A$.

[18] Note that there is no Reflexivity axiom $A \approx A$ since it would be redundant: any instance can be derived using, for example, Collapse(¬¬) in conjunction with Symmetry and Transitivity.

For f-inclusive contents, which are closed under fusion, the converse is also true: every truthmaker of $A \wedge A$ is already a truthmaker of A. To obtain the *f-inclusive system* $\mathfrak{D}_f$, we therefore add to $\mathfrak{D}_e$ the *f-inclusive collapse* axiom for $\wedge$:

FCollapse($\wedge$) $\quad A \wedge A \rightarrow A$

ECollapse($\wedge$) and FCollapse($\wedge$) are jointly equivalent, in the presence of the other axioms and rules, to the simpler axiom

Collapse($\wedge$) $\quad A \wedge A \approx A$

(This is axiom E2 in (Fine, 2016).) The latter can be derived from the former by means Commutativity($\vee$), Transitivity, and Symmetry. Both FCollapse($\wedge$) and ECollapse($\wedge$) may be derived from Collapse($\wedge$) by means of Transitivity, Symmetry, Collapse($\vee$) and Preservation($\vee$).

Turning to c-inclusive contents, we note that every c-inclusive truthmaker of $A \wedge B$ is a c-inclusive truthmaker of $A \vee (A \wedge B \wedge C)$. As a result, under the c-inclusive interpretation of $\approx$, the following axiom is valid:

Convexity $\quad (A \wedge B) \rightarrow A \vee (A \wedge (B \wedge C))$

Adding this axiom to $\mathfrak{D}_e$ results in the *c-inclusive system* $\mathfrak{D}_c$.

Finally, the system obtained by adding both FCollapse($\wedge$) and Convexity to $\mathfrak{D}_e$ will be called the *regular system* $\mathfrak{D}_r$. Within $\mathfrak{D}_r$, we can derive the second distributivity rule, or E11 in the nomenclature of (Fine, 2016):

Distributivity($\vee/\wedge$) $\quad A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

Proposition 3. $\vdash_r A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

Proof. We first establish

(*) $\vdash_r A \vee B \approx (A \vee B) \vee (A \wedge B)$

by the following derivation:[19]

$A \vee B \approx (A \vee B) \wedge (A \vee B)$	by Collapse($\wedge$)
$\approx ((A \vee B) \wedge A) \vee ((A \vee B) \wedge B)$	by Distributivity($\wedge/\vee$)
$\approx (A \wedge A) \vee (A \wedge B) \vee (A \wedge B) \vee (B \wedge B)$	by Distributivity($\wedge/\vee$)
$\approx A \vee B \vee (A \wedge B)$	by Collapse($\wedge$)

Then our result may be established as follows:

[19] Here and in what follows, we often apply the commutativity and associativity rules as well as Positive Replacement tacitly, and we drop brackets that only disambiguate between formulas equivalent by the associativity rules. That is, we write $A \vee B \vee C$ in place of $(A \vee B) \vee C$ or $A \vee (B \vee C)$, and similarly in other cases.

$(A \vee B) \wedge (A \vee C)$	
$\approx (A \wedge A) \vee (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$	by rep. appl. of Distributivity($\wedge/\vee$)
$\approx A \vee (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$	by Collapse($\wedge$)
$\approx A \vee (A \wedge B) \vee (A \wedge C) \vee (A \wedge B \wedge C) \vee (B \wedge C)$	by (*), Collapse($\wedge$)
$\approx A \vee (A \wedge B \wedge C) \vee (B \wedge C)$	by Convexity
$\approx A \vee (B \wedge C)$	by (*)

This completes the proof. □

Adding just Collapse($\wedge$) and Distributivity($\vee/\wedge$) to $\mathfrak{D}_b$ yields the set of axioms and rules that Fine uses in his 2016. It follows from Fine's completeness result relative to r-equivalence together with the soundness results below that this axiomatization is equivalent to $\mathfrak{D}_r$. As an axiomatization of the logic of r-equivalence, the Finean version is undoubtedly more natural and elegant. The advantage of the present system $\mathfrak{D}_r$ is the straightforward correlation between, on the one hand, the closure properties of closure under fusion and convexity on contents, and, on the other hand, the characteristic axioms FCollapse($\wedge$) and Convexity to which they give rise.

4.2. **Soundness and Independence.** We shall now prove that the systems $\mathfrak{D}_e$, $\mathfrak{D}_c$, $\mathfrak{D}_f$, and $\mathfrak{D}_r$ are *sound* relative to the pertinent interpretations of $\approx$.

Theorem 4. *(**Soundness**) For all equivalences $A \approx B \in \mathcal{L}_{\approx}$,*

(1) if $\vdash_e A \approx B$ then $\models_e A \approx B$
(2) if $\vdash_c A \approx B$ then $\models_c A \approx B$
(3) if $\vdash_f A \approx B$ then $\models_f A \approx B$
(4) if $\vdash_r A \approx B$ then $\models_r A \approx B$

Proof. Note first that since the f-inclusive, c-inclusive, and regular contents of a formula are functions of its exact content, sameness of exact content implies sameness of f-inclusive, c-inclusive, and regular content, and hence $\models_e A \approx B$ implies $\models_c A \approx B$, $\models_f A \approx B$, and $\models_r A \approx B$.

For (1):

- The validity of the Symmetry and Transitivity rules is immediate from the fact that $\approx$ is interpreted by an equivalence relation.
- The validity of the Preservation rules is immediate from the fact that, by definition of $\wedge$ and $\vee$, the positive content of a disjunction and a conjunction is a function of the positive content of the disjuncts and conjuncts, respectively.
- The validity of Collapse($\vee$), Commutativity($\vee$) and Associativity($\vee$) follows from the idempotence, commutativity and associativity of set-theoretic union.

- The validity of Commutativity($\wedge$) and Associativity($\wedge$) is readily verified by noting that the binary fusion operation on states is commutative and associative.
- For Distributivity($\wedge/\vee$), suppose first that $s \in [A \wedge (B \vee C)]^+$. Then $s = t \sqcup u$ for some t, u with $t \in [A]^+$ and $u \in [B \vee C]^+$, hence either (i) $u \in [B]^+$ or (ii) $u \in [C]^+$. If (i), then $t \sqcup u \in [A \wedge B]^+$. If (ii), then $t \sqcup u \in [A \wedge C]^+$. Either way, $s = t \sqcup u \in [A \wedge B]^+ \cup [A \wedge C]^+ = [(A \wedge B) \vee (A \wedge C)]^+$. Suppose now that $s \in [(A \wedge B) \vee (A \wedge C)]^+ = [A \wedge B]^+ \cup [A \wedge C]^+$, hence either (i) $s \in [A \wedge B]^+$ or (ii) $s \in [A \wedge C]^+$. If (i), then $s = t \sqcup u$ for some $t \in [A]^+$ and $u \in [B]^+$. But then $u \in [B \vee C]^+$, hence $s \in [A \wedge (B \vee C)]^+$. If (ii), then $s = t \sqcup u$ for some $t \in [A]^+$ and $u \in [C]^+$. Then again, $u \in [B \vee C]^+$, hence $s \in [A \wedge (B \vee C)]^+$.
- Collapse($\neg\neg$): $[\neg\neg A]^+ = [\neg A]^- = [A]^+$ by the definition of negation.
- DeMorgan($\neg\wedge$): $[\neg(A \wedge B)]^+ = [A \wedge B]^- = [A]^- \vee [B]^- = [\neg A]^+ \vee [\neg B]^+ = [\neg A \vee \neg B]^+$
- DeMorgan($\neg\vee$): $[\neg(A \vee B)]^+ = [A \vee B]^- = [A]^- \wedge [B]^- = [\neg A]^+ \wedge [\neg B]^+ = [\neg A \wedge \neg B]^+$
- For ECollapse($\wedge$), note that by idempotence of state-fusion, $[A]^+ \subseteq [A \wedge A]^+$.

For (2), we need to establish the validity of Convexity relative to the c-inclusive interpretation of $\approx$, i.e. to show that $[A \wedge B]^+_c \subseteq [A \vee (A \wedge B \wedge C)]^+_c$. So suppose $t \in [A \wedge B]^+_c$, and hence $s \sqsubseteq t \sqsubseteq u$ for some $s, u \in [A \wedge B]^+$. Then $s^- \in [A]^+$ and hence $s^- \in [A \vee (A \wedge B \wedge C)]^+$ for some $s^- \sqsubseteq s$. Moreover, $u \sqsubseteq u^+$ for some $u^+ \in [A \vee (A \wedge B \wedge C)]^+$. Since $s^- \sqsubseteq t \sqsubseteq u^+$, it follows that $t \in [A \vee (A \wedge B \wedge C)]^+_c$.

For (3), we also need to establish the validity of FCollapse($\wedge$) relative to the f-inclusive interpretation of $\approx$. It suffices to show that $[A \wedge A]^+_f \subseteq [A]^+_f$. So suppose $s \in [A \wedge A]^+_f$, and hence $s = \bigsqcup T$ for some $T \subseteq [A \wedge A]^+$. We may associate each $t \in T$ with states $t_1, t_2 \in [A]^+$ with $t = t_1 \sqcup t_2$. Let $T^* = \bigcup\{\{t_1, t_2\} : t \in T\}$. Then $T^* \subseteq [A]^+$, and hence $\bigsqcup T^* \in [A]^+_f$. But by associativity of fusion, $\bigsqcup T = \bigsqcup T^*$, and hence $s = \bigsqcup T^*$, so $s \in [A]^+_f$, as desired.

For (4), we need to establish the validity of FCollapse($\wedge$) and Convexity relative to the regular interpretation of $\approx$. So we need to show, firstly, that $[A \wedge A]^+_r \subseteq [A]^+_r$. So suppose $t \in [A \wedge A]^+_r$, and hence $s \sqsubseteq t \sqsubseteq \bigsqcup[A \wedge A]^+$ for some $s \in [A \wedge A]^+$. Then $s = s_1 \sqcup s_2$ for some $s_1, s_2 \in [A]^+$. By associativity of fusion, $\bigsqcup[A \wedge A]^+ = \bigsqcup[A]^+$, so $s_1 \sqsubseteq t \sqsubseteq \bigsqcup[A]^+$, and since $s_1 \in [A]^+$, it follows that $t \in [A]^+_r$. We need to show, secondly, that $[A \wedge B]^+_r \subseteq [A \vee (A \wedge B \wedge C)]^+_r$. So suppose $t \in [A \wedge B]^+_r$, and hence $s \sqsubseteq t \sqsubseteq \bigsqcup[A \wedge B]^+$ for some $s \in [A \wedge B]^+$. Then some part s_A of s is in $[A]^+$ and hence in $[A \vee (A \wedge B \wedge C)]^+$. Since $\bigsqcup[A \wedge B]^+ \sqsubseteq \bigsqcup[A \vee (A \wedge B \wedge C)]^+$, it follows that $s_A \sqsubseteq s \sqsubseteq \bigsqcup[A \vee (A \wedge B \wedge C)]^+$, and hence that $s \in [A \vee (A \wedge B \wedge C)]^+_r$. □

It is known for the Finean system, which is obtained by adding both Collapse($\wedge$) and Distributivity($\vee/\wedge$) to $\mathfrak{D}_b$, that its axioms and rules are independent of one another (cf. (Fine, 2016: p. 201n3)). We can show that the same is true for $\mathfrak{D}_e$ by proving that ECollapse($\wedge$) is not derivable within $\mathfrak{D}_b$.[20] To do this, we give an alternative interpretation to the language $\mathcal{L}_\approx$ under which $\mathfrak{D}_b$ is sound but ECollapse($\wedge$) is not. The interpretation will make crucial use of multi-sets, which are like sets but may contain each member any finite number of times. If S is a multi-set, let $\#(S, x)$ be the number of occurrences of x in S. Given multi-sets S and T, let their *sum* $S + T$ be the multi-set U such that $\#(U, x) = \#(S, x) + \#(T, x)$ for all x. We write $|x, x, \ldots, y, \ldots|$ for the multi-set containing each of $x, y, \ldots$ exactly as many times as it is listed.

Definition 12. *A* multi-model *of $\mathcal{L}_\approx$ is any pair $(S, [\cdot])$ where S is a non-empty set of multi-sets closed under summation, and $[\cdot]$ maps every sentence letter p of $\mathcal{L}$ to a pair $([p]^+, [p]^-)$ of non-empty subsets of S.*

Relative to a given multi-model, we extend $[\cdot]$ to all formulas of $\mathcal{L}$ as follows:

- $[A \vee B] = ([A]^+ \cup [B]^+, \{s + t : s \in [A]^- \text{ and } t \in [B]^-\})$
- $[A \wedge B] = (\{s + t : s \in [A]^+ \text{ and } t \in [B]^+\}, [A]^- \cup [B]^-)$
- $[\neg A] = ([A]^-, [A]^+)$

In effect, rather than interpreting conjunction in terms of *fusions* of verifiers, we now interpret in terms of *sums*, in the multi-set sense just defined, of verifiers.

Proposition 5. *ECollapse($\wedge$) is not derivable within $\mathfrak{D}_b$.*

Proof. It is straightforward to verify that $\mathfrak{D}_b$ is sound relative to the class of multi-models of $\mathcal{L}_\approx$, when $A \approx B$ is taken as true iff $[A]^+ = [B]^+$. Pick a multi-model including a state $|x|$. Then pick a sentence letter p from $\mathcal{L}$ and consider a multi-model with $[p]^+ = \{|x|\}$. Then $[p \wedge p]^+ = \{|x, x|\}$ whereas $[p \vee (p \wedge p)]^+ = \{|x|, |x, x|\}$, and so $p \vee (p \wedge p) \approx p \wedge p$ is false. Since this is an instance of ECollapse($\wedge$), by soundness of $\mathfrak{D}_b$, it follows that ECollapse($\wedge$) is not derivable within $\mathfrak{D}_b$. □

We may establish parallel independence results for FCollapse($\wedge$) and Convexity.

Proposition 6. *FCollapse($\wedge$) is not derivable within $\mathfrak{D}_e$ and not derivable within $\mathfrak{D}_c$.*

Proof. Let x and y be distinct objects and consider the state-space $(\wp(\{x, y\}), \subseteq)$. Pick a sentence letter p and a model with $[p]^+ = \{\{x\}, \{y\}\}$ and thus $[p \wedge p]^+ = \{\{x\}, \{y\}, \{x, y\}\}$. Note that $[p]^+_c = [p]^+$ and likewise $[p \wedge p]^+_c = [p \wedge p]^+$. Since $[p \wedge p]^+$ is not a subset

[20] (Fine, 2018a: p. 625) states, in effect, that $\mathfrak{D}_b$ is complete with respect to exact truthmaker equivalence. It follows from Proposition 5, together with the soundness theorem 4, that this is a mistake.

of $[p]^+$, it follows that $p \wedge p \to p$ is false under the interpretations of $\approx$ as exact or c-inclusive truthmaker equivalence. Together with the soundness theorem, this establishes our claim. □

Proposition 7. *Convexity is not derivable within $\mathfrak{D}_e$ and not derivable within $\mathfrak{D}_f$.*

Proof. Let x, y, z be pairwise distinct and consider the state-space $(\wp(\{x, y, z\}), \subseteq)$. Pick distinct sentence letters p, q, r of $\mathcal{L}$ and a model with $[p]^+ = \{\{p\}\}$, $[q]^+ = \{\{q\}\}$ and $[r]^+ = \{\{r\}\}$. Then $[p \wedge q]^+_f = [p \wedge q]^+ = \{\{p, q\}\}$ and $[p \vee (p \wedge q \wedge r)]^+_f = [p \vee (p \wedge q \wedge r)]^+ = \{\{p\}, \{p, q, r\}\}$. Since $[p \wedge q]^+$ is not a subset of $[p \vee (p \wedge q \wedge r)]^+$, $p \wedge q \to p \vee (p \wedge q \wedge r)$ is false under both the exact and the f-inclusive truthmaker equivalence interpretation of $\approx$. Then by the soundness theorem, our claim follows. □

5. Disjunctive Normal Forms

Our completeness proofs are adaptations of Fine's (2016) completeness proof for regular truthmaker equivalence to the other interpretations of equivalence. They proceed via disjunctive normal forms. For each of our notions of truthmaker equivalence, we shall identify a particular class of disjunctive normal forms in $\mathcal{L}$ with the following properties. Firstly, any two distinct members of the class fail to be truthmaker equivalent in some model. Secondly, any sentence of $\mathcal{L}$ is provably equivalent to some member of the class. From these results, it is straightforward to establish completeness.[21]

As usual, we call any sentence letter and its negation a *literal* of $\mathcal{L}$. Any formula obtainable from literals purely by successive applications of $\wedge$ is called a conjunctive form. As a limiting case, a literal is also considered a *conjunctive form*. When I speak of the conjuncts of a conjunctive form, I mean only the literals that occur as conjuncts in the conjunctive form. Any formula obtainable from conjunctive forms purely by successive applications of $\vee$ is called a *disjunctive form*. As a limiting case, a conjunctive form is also considered a disjunctive form. When I speak of the disjuncts of a disjunctive form, I mean only the conjunctive forms that occur as disjuncts in the disjunctive form.

Given a conjunctive form A, let $lit(A)$ be the multi-set of literals of $\mathcal{L}$ that includes each literal of $\mathcal{L}$ exactly as many times as it occurs as a conjunct in A. (We count a

[21] Note that just as in Fine (2016) and Correia (2016), the completeness results established here are weak completeness results: the deductive systems are shown to prove all the logically true equivalences. There is also a natural way to define a notion of logical consequence within our semantics, and a different kind of deductive system would be required in order to derive all the logical consequences from a given set of premises. Although it would certainly be of interest to develop such systems for our notions of equivalence, doing so is beyond the scope of this paper. It may be worth noting that the applications of a logic of equivalence sketched in the introduction to this paper mainly seem to depend on weak completeness.

literal as occcuring as a conjunct in itself, so $lit(A) = |A|$ if A is a literal.) Note that by the Associativity and Commutativity rules for $\wedge$, in all of our deductive systems, two conjunctive forms A and B are provably equivalent – that is, $A \approx B$ is derivable – if $lit(A) = lit(B)$. We now pick some function cf that maps each multi-set X of literals of $\mathcal{L}$ to a conjunctive form $cf(X)$ with $lit(cf(X)) = X$, and denote the image of the cf-function SCF, the set of standard conjunctive forms.

Given a disjunctive form A, let $v(A)$ be $\{lit(B) : B \text{ is a disjunct in } A\}$. (We count a conjunctive form as occurring as a disjunct in itself, so $v(A) = \{lit(A)\}$ if A is a conjunctive form.) By the Associativity and Commutativity rules for $\vee$ and $\wedge$, in all of our deductive systems, two disjunctive forms A and B are provably equivalent if $v(A) = v(B)$. We now pick some function df mapping each set M of multi-sets of literals of $\mathcal{L}$ some disjunctive form $df(M)$ such that (i) $v(df(M)) = M$, (ii) no disjunct in $df(M)$ occurs more than once, and (ii) all of the disjuncts in M are standard conjunctive forms. We denote the image of the df-function SDF, the set of standard disjunctive forms.

Lemma 8. *Every sentence in $\mathcal{L}$ is provably equivalent to a disjunctive form.*

Proof. This is a straightforward corollary of Theorem 15 in (Fine, 2016: p. 214). □

We say that sentences A and B are *strictly equivalent* within a given deductive system iff A and B are provably equivalent within the deductive system and $v(A) = v(B)$. Then

Lemma 9. *Every disjunctive form is strictly equivalent to a standard disjunctive form.*

Proof. By standard methods using Collapse($\vee$), Commutativity($\vee$), Associativity($\vee$), Commutativity($\wedge$), Associativity($\wedge$) and PR. □

5.1. **Exact Truthmaking.** We now wish to specify a subclass of SDF with the two desired properties mentioned above, that any sentence in $\mathcal{L}$ is provably equivalent, within $\mathfrak{D}_e$, to a member of the class, and that any two distinct members of the class can differ with respect to their exact truthmakers. Note that SDF itself does not have the second property. For let A be some literal and consider the formulas $A \wedge A$ and $A \vee (A \wedge A)$. By the soundness of ECollapse($\wedge$) for exact truthmaker equivalence, we know that these have the same exact truthmakers in every model. But note that $v(A \wedge A) = \{|A, A|\}$ and $v(A \vee (A \wedge A)) = \{|A|, |A, A|\}$. Now consider the standard disjunctive form $B = df(v(A \wedge A))$ and the standard disjunctive form $C = df(v(A \vee (A \wedge A)))$. Since $v(B) = v(A \wedge A)$, B is provably equivalent to $A \wedge A$, and since $v(C) = v(A \vee (A \wedge A))$, C is provably equivalent to $A \vee (A \wedge A)$, so B is provably equivalent to C, and hence B and C have the same exact truthmakers in every model. But B and C are distinct members of SDF, since $v(B) = v(A \wedge A) = \{|A, A|\} \neq \{|A|, |A, A|\} = v(A \vee (A \wedge A)) = v(C)$.

So from any subclass of SDF whose members are pairwise related relevantly like $A \wedge A$ and $A \vee (A \wedge A)$ are, we need to pick a unique representative. To this end, we define a notion of *subsumption* between multi-sets of literals.

Definition 13. *For X and Y multi-sets of literals of $\mathcal{L}$, say that*

- *X is subsumed by Y iff (i) the same set underlies X and Y, and every literal that occurs in X occurs at least as many times in Y,*
- *X is properly subsumed by Y iff X is subsumed by Y and Y is not subsumed by X.*

By extension, we say that a conjunctive form A is (properly) subsumed by a conjunctive form B iff $lit(A)$ is (properly) subsumed by $lit(B)$. Next, we define a notion of fullness for sets of multi-sets of literals.

Definition 14. *A set M of multi-sets of literals of $\mathcal{L}$ is* full *iff every multi-set of literals properly subsumed by some member of M is itself a member of M.*

Again, by extension, we say that a disjunctive form A is full iff $v(A)$ is full.

Lemma 10. *Every disjunctive form is provably equivalent within $\mathfrak{D}_e$ to a full disjunctive form.*

Proof. Let F be a disjunctive form. Let F's fall-off value n be the number of multi-sets of literals X such that: (i) X is properly subsumed by some member of $v(F)$, and (ii) X is not itself a member of $v(F)$. If $n = 0$, then F is itself a full disjunctive form and we are done. So suppose $n > 0$. We now pick some multi-set of literals X which is not a member of $v(F)$ but which is *immediately* subsumed by some member Y of $v(F)$, i.e. X is subsumed by Y, and there is exactly one literal in Y that occurs less often in X than in Y, and it occurs exactly one time less in X than in Y. Now let B be a disjunct in F with $lit(B) = Y$. Then B and $cf(X)$ are respectively provably equivalent to conjunctive forms of the forms $(C \wedge C) \wedge D$ and $C \wedge D$. But by ECollapse($\wedge$), $C \wedge C \approx C \vee (C \wedge C)$, and so

$$\begin{aligned} B &\approx (C \wedge C) \wedge D \\ &\approx (C \vee (C \wedge C)) \wedge D && \text{by ECollapse}(\wedge)\text{, Preservation}(\wedge) \\ &\approx (C \wedge D) \vee ((C \wedge C) \wedge D) && \text{by Distributivity}(\wedge/\vee) \\ &\approx B \vee cf(X) && \text{by PR} \end{aligned}$$

By another application of PR, F is provably equivalent to the result F' of replacing B in F by $B \vee cf(X)$. The fall-off value of F' is strictly less than n. □

Lemma 11. *Every standard disjunctive form is provably equivalent, within $\mathfrak{D}_e$, to a full standard disjunctive form.*

Proof. Let F be a standard disjunctive form. By the lemma 10, F is provably equivalent to a full disjunctive form F'. By lemma 9, F' is strictly equivalent to some standard disjunctive form F^*. Since F' and F^* are strictly equivalent, $v(F') = v(F^*)$ and so since F' is full, so is F^*. □

5.2. **C-Inclusive Truthmaking.** The distinctive feature of c-inclusive equivalence is that it does not distinguish between $A \vee (A \wedge B \wedge C)$ and $A \vee (A \wedge B) \vee (A \wedge B \wedge C)$. So for proving completeness for the system $\mathfrak{D}_c$, we need to pick one representative from each class of full standard disjunctive forms that are related in this way. In order to define the relevant relationship, let us say that, for any multi-sets X and Y, X is a sub-multiset of Y ($X \subseteq_m Y$) iff every element of X occurs at least as many times in Y as it does in X. We then say that a multi-set Y is *between* multi-sets X and Z iff $X \subseteq_m Y \subseteq_m Z$. Finally, a disjunctive form A will be said to be convex iff whenever a multi-set of literals X lies between $lit(B)$ and $lit(C)$ for B and C disjuncts in A, some conjunctive form D with $X = lit(D)$ is also a disjunct in A. The class of standard disjunctive forms relevant to proving completeness will be the class of full and convex standard disjunctive forms.

Lemma 12. *Every full disjunctive form is provably equivalent, within $\mathfrak{D}_c$, to a full and convex disjunctive form.*

Proof. Let A be a full disjunctive form. Let A's fall-off value n be the number of multi-sets X witnessing A's non-convexity, i.e. that lie between $lit(B)$ and $lit(C)$ for some disjuncts B and C of A, while there is no disjunct D in A with $lit(D) = X$. If $n = 0$, then A is already full and convex. So suppose $n > 0$ and let B and C be a witnessing pair of disjuncts in A. Note that B is not subsumed by C, for otherwise, by fullness of A, there would be a disjunct D with $lit(D) = X$ for every multi-set X between $lit(B)$ and $lit(C)$. So there is some literal q that occurs at least once in C but not at all in B.

Thus B and C are provably equivalent, respectively, to conjunctive forms of the forms E and $E \wedge q \wedge F$. Using PR, A may be shown to be provably equivalent to a disjunctive form of the form $E \vee (E \wedge q \wedge F) \vee G$. By the Convexity rule of $\mathfrak{D}_c$, $E \vee (E \wedge q \wedge F)$ is provably equivalent to $E \vee (E \wedge q) \vee (E \wedge q \wedge F)$, and thus A is provably equivalent to $A' = E \vee (E \wedge q) \vee (E \wedge q \wedge F) \vee G$. The fall-off value of A' is then less than n.

Now A' may no longer be full, since there may be multi-sets Y subsumed by $lit(E \wedge q)$ for which there is no corresponding disjunct in A'. But by the methods used in the proof of lemma 10, A' may be expanded to a full disjunctive form A^* that is provably equivalent to A'. It remains to show that the fall-of value of A^* is still less than n. To this end, note first that for any disjunct G added in forming A^*, $lit(G) \subseteq_m lit(C)$. Now say that a conjunctive form A is *repetition-free* iff no literal occurs more than once as a conjunct in A. Then by fullness of A, there is a repetition-free counterpart B^* of B that occurs in A, and hence A'. We then note, secondly, that for any disjunct G added in forming A^*, $lit(B^*) \subseteq_m lit(G)$. It follows that any multi-set X witnessing the non-convexity of A^* also witnesses the non-convexity of A'. So the fall-off value of A^* is less than or equal to that of A', and hence less than n. □

Lemma 13. *Every standard disjunctive form is provably equivalent, within $\mathfrak{D}_c$, to a full and convex standard disjunctive form.*

Proof. Let A be a standard disjunctive form. By the previous lemma, A is provably equivalent to a full and convex disjunctive form A'. By lemma 9, A' is strictly equivalent to a standard disjunctive form A^*. Since $v(A') = v(A^*)$ by strict equivalence, A^* is full and convex. □

5.3. **F-Inclusive Truthmaking.** When we turn to the notion of f-inclusive equivalence, two things change in comparison to exact equivalence.[22] Firstly, we no longer need to distinguish between conjunctive forms one of which subsumes the other: A will be equivalent to $A \wedge A$, and similarly in other cases. Secondly, we no longer need to distinguish between disjunctive forms related like $A \vee B$ and $A \vee B \vee (A \wedge B)$, where one contains some extra disjuncts, all of which are however conjunctions of disjuncts present in the other. So let us say firstly, that a disjunctive form is repetition-free iff every one of its disjuncts is. Secondly, let us say that a disjunctive form A is *closed* iff for any two disjuncts B and C in A, A includes a disjunct D with $slit(D) = slit(B) \cup slit(C)$, where $slit(X)$ is the set of literals occurring as conjuncts in the conjunctive form X. Note that since any disjunctive form has only finitely many disjuncts, closure of A implies that for any collection Γ of disjuncts in A, A has a disjunct D with $slit(D) = \bigcup\{slit(B) : B \in \Gamma\}$. The subclass of standard disjunctive forms relevant for f-inclusive equivalence will that of the closed and repetition-free standard disjunctive forms.

Lemma 14. *Every disjunctive form is provably equivalent, within $\mathfrak{D}_f$, to a closed disjunctive form.*

Proof. We show first that any instance of $A \vee B$ is equivalent to the corresponding instance of $A \vee B \vee (A \wedge B)$.

$$
\begin{aligned}
A \vee B &\approx (A \vee B) \wedge (A \vee B) && \text{by FCollapse}(\wedge) \\
&\approx ((A \vee B) \wedge A) \vee ((A \vee B) \wedge B) && \text{by Distributivity}(\wedge/\vee)
\end{aligned}
$$

[22] The completeness of $\mathfrak{D}_f$ with respect to f-inclusive equivalence has already been established in Correia (2016). Since Correia's proof proceeds in a slightly different way than the ones given here, I'm giving another proof that proceeds in exact parallel to those for exact and c-inclusive equivalence.

$$\begin{aligned} &\approx (A \wedge A) \vee (A \wedge B) \vee (A \wedge B) \vee (B \wedge B) && \text{by Distributivity } (\wedge/\vee) \\ &\approx A \vee B \vee (A \wedge B) \vee (A \wedge B) && \text{by FCollapse}(\wedge) \\ &\approx A \vee B \vee (A \wedge B) && \text{by Collapse}(\vee) \end{aligned}$$

Now let A be a disjunctive form, and let A's fall-off value n be the number of collections Γ of disjuncts in A for which there is no disjunct D in A with $slit(D) = \bigcup\{slit(B) : B \in \Gamma\}$. We prove the result by induction on n. If $n = 0$, A is already closed. Suppose $n > 0$. Since there are only finitely many disjuncts in A, there will be some pair $\{B, C\}$ that satisfies this condition. Using the above equivalence and PR as well as appropriate reshuffling of disjuncts using Associativity($\vee$) and Commutativity($\vee$), we can derive the equivalence of A and $A \vee (B \wedge C)$. The fall-off value of $A \vee (B \wedge C)$ is strictly less than n. □

Lemma 15. *Every conjunctive form is provably equivalent, within $\mathfrak{D}_f$, to repetition-free conjunctive form.*

Proof. By FCollapse($\wedge$) and PR, using Associativity($\wedge$) and Commutativity($\wedge$) to shuffle around the conjuncts so as to put repeat occurrences of the same literal next to each other. □

Lemma 16. *Every disjunctive form is provably equivalent, within $\mathfrak{D}_f$, to a repetition-free disjunctive form.*

Proof. By the previous lemma and PR. □

Lemma 17. *Every standard disjunctive form is provably equivalent, within Df, to a closed and repetition-free standard disjunctive form.*

Proof. Let A be a standard disjunctive form. By lemma 14, it is equivalent to closed disjunctive form A'. By PR and lemma 15, every disjunct in A' may be replaced by a repetition-free counterpart, yielding a repetition-free disjunctive form A''. Note that the replacement preserves closure, since whenever B' is a repetition-free counterpart of B, $slit(B') = slit(B)$. So A'' is repetition-free and closed. By lemma 9, A'' is strictly equivalent to a standard disjunctive form A'''. Since both repetition-freedom and closure of a disjunctive form X depend only on $v(X)$, and $v(A'') = v(A''')$ by strict equivalence, A''' is a closed and repetition-free standard disjunctive form, as required. □

5.4. **Regular Truthmaking.** This is the case that is covered in Fine (2016), so I shall merely mention very briefly the main definitions and results; for details I refer the reader to section 7 of Fine's paper. The relevant subclass of standard disjunctive forms comprises those that Fine calls *maximal*, which are defined as follows. A disjunctive form A is maximal iff for every disjunct D in A and every literal q occurring as a conjunct in a disjunct of A, A contains a disjunct D' with $slit(D') = slit(D) \cup \{q\}$. Using similar techniques to those we have employed above, Fine shows that every disjunctive form is provably equivalent, within $\mathfrak{D}_r$, to a maximal disjunctive form. The crucial auxiliary result is that the Distributivity($\vee/\wedge$) rule in $\mathfrak{D}_r$ allows us to prove the equivalence of $A \vee (B \wedge C)$ to $A \vee (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$, the latter being the obvious maximal counterpart to the former. By the usual means, it may then be established that

Lemma 18. *Every standard disjunctive form is provably equivalent, within $\mathfrak{D}_r$, to a maximal standard disjunctive form.*

6. Canonical Models

In this section, we define, for each of our notions of truthmaker-equivalence, a suitable canonical model to prove completeness, i.e. a model in which no two members of the relevant class of disjunctive normal forms are assigned the same exact, c-inclusive, f-inclusive, or regular content, depending on the notion of equivalence in question.

6.1. **Exact Truthmaking.** The simplest strategy for defining a canonical model is to let each sentence letter A of $\mathcal{L}$ be verified by A and only A, and falsified by $\neg A$ and only $\neg A$. But as a canonical model for exact truthmaker equivalence, this will not do. For it is a straightforward consequence of the definition of conjunction that any sentence A with only a single verifier will have the same set of verifiers as its self-conjunction $A \wedge A$. So the simple canonical model will not allow us to invalidate all desired instances of $A \approx A \wedge A$. For example, whenever A is a literal, $A \approx A \wedge A$ will be true in the simple canonical model. To avoid this problem, we may instead associate each sentence letter A with a countable infinity of verifiers, to make sure that the result of conjoining A with itself any finite number of times will always produce some new verifier.

Definition 15. *The* exact canonical model $\mathcal{M}_E$ *of $\mathcal{L}$ is the triple* $(S, \sqsubseteq, [\cdot])$ *with*

- $S = \wp(\{(A, i) : A \text{ is a literal of } \mathcal{L} \text{ and } i \in \mathbb{N}\})$
- $\sqsubseteq$ *is the restriction of subsethood to S*
- $[p] = (\{\{(p, i)\} : i \in \mathbb{N}\}, \{\{(\neg p, i)\} : i \in \mathbb{N}\})$

From the fact that S is a powerset and $\sqsubseteq$ the restriction of the subset-relation to that powerset, it is straightforward that $(S, \sqsubseteq)$ is a state-space and hence that the exact canonical model is indeed a model of $\mathcal{L}$.

Lemma 19. *In $\mathcal{M}_E$, for any conjunctive form A and literal q, if q occurs a total of n times as a conjunct in A, then*

(1) every verifier of A has at most n pairs of the form (q, i) as members, and

(2) *some verifier of A has n distinct pairs of the form* (q, i) *as members, and*
(3) *every verifier of A has at least one pair of the form* (q, i) *as member, provided* $n > 0$.

Proof. Part (3) is immediate from the definition of conjunction and the fact that by construction of the model, every verifier of q is of the form $\{(q, i)\}$.

Parts (1) and (2) may be proved by induction on n. Case $n = 0$: Part (1) is immediate from the definition of conjunction and the fact that by construction of $\mathcal{M}_E$, no literal except q has a verifier with some pair of the form (q, i) as a member. Part (2) is immediate from part (1) and the fact that A has a verifier, which is clear by construction of the model.

Case $n = m + 1$: Without loss of generality, assume that A is of the form $B \wedge q$. Then B contains a total of m occurrences of q, so by IH, every verifier B contains at most m pairs of the form (q, i) as members, and some verifier of B contains m pairs of that form. For part (1), suppose s verifies A. Then $s = t \cup u$ for some t verifying B and some u verifying q. By construction of the canonical model, $u = \{(q, j)\}$ for some j. Since t contains at most m pairs of the form $\{(q, i)\}$, s contains at most $m + 1 = n$ pairs of that form. For part (2), by IH, we may pick a verifier t of B with m pairs of the form (q, i). Let $j = max\{i \in \mathbb{N} : (q, i) \in t\}$. Then $\{(q, j + 1)\}$ verifies q, and hence $s = t \cup \{(q, j + 1)\}$ verifies A. By definition of j, $(q, j + 1)$ is not a member of t, so s contains $m + 1 = n$ pairs of the form (q, i). □

Lemma 20. *Let A be a conjunctive form. In* $\mathcal{M}_E$, *there is some verifier s of A such that every conjunctive form verified by s subsumes A.*

Proof. Let s be a verifier of A as per part (2) of the previous lemma. So for any literal q occurring $n \geq 1$ times in A, s has n distinct pairs of the form (q, i) as member. Suppose s verifies a conjunctive form B. Then by part (1) of the previous lemma, each literal q occurring in A occurs at least as often in B. It remains to show that the same set underlies $lit(A)$ and $lit(B)$. We have already seen that every literal occurring in A also occurs in B, so it remains only to show that every literal occurring in B occurs at least once in A. So suppose q occurs in B. Then by part (3) of the previous lemma, every verifier of B, and therefore s, contains at least one pair of the form (q, i) as a member. But then by part (1) of the previous lemma, q occurs as a conjunct in A. □

Theorem 21. *In* $\mathcal{M}_E$, *if A and B are distinct full standard disjunctive forms of* $\mathcal{L}$, *then* $[A]^+_e \neq [B]^+_e$.

Proof. Suppose A and B are distinct full standard disjunctive forms. Without loss of generality, assume there is a disjunct D in A which does not occur in B. By the previous lemma, there is a verifier s of D, and hence of A, which verifies only conjunctive forms that subsume D. Now suppose for reductio that s verifies B. By the definition of disjunction, s verifies some disjunct E of B. Since s verifies only conjunctive forms subsuming D, E subsumes D. Since B is a full standard disjunctive form, $v(B)$ is full. Since $v(B)$ has $lit(E)$ as a member, and $lit(E)$ subsumes $lit(D)$, by fullness, $v(B)$ also has $lit(D)$ as a member. Since B is a standard disjunctive form, B includes $cf(lit(D))$ as a disjunct. But since D is a standard conjunctive form, $D = cf(lit(D))$, so B includes D as a disjunct, contrary to our assumption. So s does not verify B, and hence $[A]^+_e \neq [B]^+_e$. □

Note that for any given pair A and B of distinct full standard disjunctive forms of $\mathcal{L}$, we can also find a *finite* model in which they have different verifiers. For any literal p and formula F of $\mathcal{L}$, let $o(p, F)$ the number of occurrences of p in F. Now let $S_0 = \{(p, i) : p$ is a literal that occurs in A or in B and $i \leq max\{o(p, A),\ o(p, B)\}\}$. Then in place of $\mathcal{M}_E$, we can pick any model with state-space $(\wp(S_0), \subseteq)$ in which any literal p in A and B is interpreted as verified by exactly the states of the form $\{(p, i)\}$. Since only finitely many literals occur in A and B, and since they occur only a finite number of times, any such model is finite. For the components of A and B, the proofs of lemmas 19 and 20, and thus of theorem 21 carry over unchanged to any such model. It follows that the logic of exact equivalence has the finite model property and therefore is decidable.

6.2. **C-Inclusive Truthmaking.** The same canonical model that we used in the previous section for exact truthmaker equivalence is also suitable for proving completeness for c-inclusive equivalence.

Definition 16. *The* c-inclusive canonical model $\mathcal{M}_C$ *for* $\mathcal{L}$ *is* $\mathcal{M}_E$.

Recall that $\#(X, x)$ is the number of occurrences of x in the multi-set X. Now for any state s in $\mathcal{M}_C$, let X_s be that multi-set of literals such that for every literal q, $\#(X_s, q)$ equals the number of distinct pairs of the form (q, i) in s. Then

Lemma 22. *For every state s and conjunctive form A,*

(1) if $lit(A) = X_s$ then s verifies A
(2) if s verifies A, then $X_s \subseteq_m lit(A)$
(3) if s verifies A and A is repetition-free, then $X_s = lit(A)$

Proof. Part (1): By induction on the complexity of A. If A is a literal, then $X_s = lit(A)$ iff $s = \{(A, i)\}$ for some $i \in \mathbb{N}$. By definition of the canonical model, any such s verifies A. Now suppose that $A = B \wedge C$ and that the claim holds for B and C (IH). Suppose s is such that $X_s = lit(A) = lit(B \wedge C)$. We need to show that s verifies $B \wedge C$. To this end, we aim to construct states t and u with $s = t \sqcup u$ such that t verifies B and u verifies C. Since by (IH), the claim holds for B and C, it suffices to show that $X_t = lit(B)$ and $X_u = lit(C)$. Now for any literal q, $\#(lit(B \wedge C), q) = \#(lit(B), q) + \#(lit(C), q)$. Since $X_s = lit(B \wedge C)$, we know that for any literal q occurring in $lit(B \wedge C)$, s contains $\#(lit(B), q) + \#(lit(C), q)$ distinct pairs of the form (q, i) as members. So we pick $\#(lit(B), q)$ one of them as members of t and the other $\#(lit(C), q)$ as members of u. Then $s = t \cup u = t \sqcup u$. Moreover, by construction of t and u, $X_t = lit(B)$ and $X_u = lit(C)$, and so by (IH), t verifies B and u verifies C, hence s verifies $A = B \wedge C$.

Part (2): By induction on the complexity of A. If A is a literal, then by construction of the canonical model, $s = \{(A, i)\}$ for some $i \in \mathbb{N}$, so $X_s = |A| = lit(A)$. Suppose $A = B \wedge C$ and the claim holds for B and C. Suppose s verifies A. Then $s = t \sqcup u$ for some t, u verifying B, C, respectively. By induction hypothesis, $X_t \subseteq_m lit(B)$ and $X_u \subseteq_m lit(C)$. Then since $s = t \cup u$, for all q, $\#(X_s, q) \leq \#(X_t, q) + (X_u, q)$. Since $X_t \subseteq_m lit(B)$, we have $\#(X_t, q) \leq \#(lit(B), q)$ and since $X_u \subseteq_m lit(C)$, we have $\#(X_u, q) \leq \#(lit(C), q)$. So for all q, $\#(X_s, q) \leq \#(lit(B), q) + \#(lit(C), q) = \#(lit(B \wedge C), q)$, and therefore $X_s \subseteq_m lit(B \wedge C)$, as desired.

Part (3): It is readily verified that if s verifies a repetition-free conjunctive form A, then s contains exactly one member of the form (q, i) for each literal occurring in A. So X_s contains each literal in A exactly once. Likewise, since A is repetition-free, $lit(A)$ contains each literal in A exactly once, and so $lit(A) = X_s$. □

We can now show that for full convex disjunctive forms, the convex truthmakers and the exact truthmakers assigned by the canonical model coincide.

Lemma 23. *In $\mathcal{M}_C$, for any full convex disjunctive form A, $[A]^+_c = [A]^+$.*

Proof. Let A be a convex disjunctive form. Since $[A]^+_c$ is the convex closure of $[A]^+$, clearly $[A]^+ \subseteq [A]^+_c$. It remains to show that $[A]^+_c \subseteq [A]^+$. So suppose $t \in [A]^+_c$. Then there are states s and u with $s \sqsubseteq t \sqsubseteq u$ and $s \in [A]^+$ and $u \in [A]^+$. By the definition of disjunction and since A is full, s verifies some repetition-free disjunct A_s in A and u verifies some disjunct A_u in A. By lemma 22(3), $X_s = lit(A_s)$ and by lemma 22(2), $X_u \subseteq_m lit(A_u)$. Since $s \sqsubseteq t \sqsubseteq u$, we have $lit(A_s) = X_s \subseteq_m X_t \subseteq_m X_u \subseteq_m lit(A_u)$, so by convexity of A, there is a disjunct D in A with $lit(D) = X_t$. By lemma 22(1), t verifies D and hence $t \in [A]^+$, as desired. □

Theorem 24. *In $\mathcal{M}_C$, if A and B are distinct, full and convex standard disjunctive forms of $\mathcal{L}$, then $[A]^+_c \neq [B]^+_c$.*

Proof. Since A and B are convex, $[A]^+_c = [A]^+$ and $[B]^+_c = [B]^+$. Since A and B are distinct, full standard disjunctive forms, it follows by theorem 21 that $[A]^+ \neq [B]^+$. □

The same method as before for restricting the canonical model to a finite one given any suitable pair A and B can be used to show that the logic of c-equivalence is decidable.

6.3. **F-Inclusive Truthmaking.** For this case, since we want to validate $A \approx A \wedge A$ and similar cases, we can work with the simple canonical model.

Definition 17. *The* f-inclusive canonical model $\mathcal{M}_F$ *for* $\mathcal{L}$ *is the triple* $(S, \sqsubseteq, [\cdot])$ *with*

- $S = \wp(\{A : A \text{ is a literal in } \mathcal{L}\})$
- $\sqsubseteq$ *is the restriction of subsethood to* S
- $[p] = (\{p\}, \{\neg p\})$

As before, since S is a powerset and $\sqsubseteq$ the restriction of the subset-relation, $(S, \sqsubseteq)$ is a state-space and hence the f-inclusive canonical model is a model of $\mathcal{L}$.

Lemma 25. *In* $\mathcal{M}_F$*, for any conjunctive form* A*,* $[A]^+ = \{slit(A)\}$.

Proof. If A is a sentence letter, then by definition of the canonical model, $[A]^+ = \{\{A\}\} = slit(A)$. If A is the negation $\neg B$ of a sentence letter B, then $[A]^+ = [B]^- = \{\{\neg B\}\} = slit(A)$. Suppose the claim holds for A and B. Then $[A \wedge B]^+ = [A]^+ \wedge [B]^+ = \{s \sqcup t : s \in [A]^+ \text{ and } t \in [B]^+\} = \{s \cup t : s \in \{slit(A)\} \text{ and } t \in \{slit(B)\}\} = \{slit(A) \cup slit(B)\} = \{slit(A \wedge B)\}$. It follows that the claim holds for all conjunctive forms. □

Lemma 26. *In* $\mathcal{M}_F$*, for any closed disjunctive form* A*,* $[A]^+_f = [A]^+$.

Proof. Since $[A]^+_f$ is the closure under fusion of $[A]^+$, we have $[A]^+ \subseteq [A]^+_f$. So suppose $s \in [A]^+_f$. Then for some $T \subseteq [A]^+$, $s = \bigsqcup T = \bigcup T$. For each $t \in T$, t verifies some conjunctive form B_t occurring as a disjunct in A, so by the previous lemma, $t = slit(B_t)$, so $s = \bigcup\{slit(B_t) : t \in T\}$. Since A is closed, some conjunctive form B with $slit(B) = \bigcup\{slit(B_t) : t \in T\} = s$ occurs as a disjunct in A. By the previous lemma, B is verified by $slit(B) = s$, so s verifies A and hence $s \in [A]^+$, as required. □

Theorem 27. *In* $\mathcal{M}_F$*, if* A *and* B *are distinct closed and repetition-free standard disjunctive forms of* $\mathcal{L}$*, then* $[A]^+_f \neq [B]^+_f$.

Proof. Suppose A and B are distinct closed and repetition-free standard disjunctive forms. Without loss of generality, assume there is a disjunct D in A which does not occur in B. By lemma 25, $[D]^+ = \{slit(D)\}$, so $slit(D)$ is in $[A]^+$ and hence $[A]^+_f$. Suppose $slit(D)$ is also in $[B]^+_f$ and thus, by the previous lemma, in $[B]^+$. By the clause for disjunction, it follows that $slit(D)$ verifies some standard conjunctive form E that occurs as a disjunct in B. But by lemma 25, $[E]^+ = \{slit(E)\}$, so it follows that $slit(D) = slit(E)$.

But since both D and E are standard conjunctive forms, it follows that $D = E$, contrary to our assumption. □

Similarly as before, decidability may be established by finding finite model to do the job of $\mathcal{M}_F$ for any given pair A and B, which here requires merely restricting the states to sets of literals occurring in A or B.

6.4. **Regular Truthmaking.** Again, this case is covered in Fine (2016), so I provide only a very brief summary; for details, readers may consult section 8 in Fine's paper. It turns out that the same, simple canonical model may here be used as was used as the f-inclusive canonical model.

Definition 18. *The* regular canonical model $\mathcal{M}_R$ *for* $\mathcal{L}$ *is* $\mathcal{M}_F$.

As shown above, a conjunctive form A has $slit(A)$ as its only verifier, and so a disjunctive form B is verified by all and only the states $slit(A)$ for A a disjunct in B. It may then be shown that for any maximal disjunctive form A, $[A]^+ = [A]^+_r$. On that basis, we can shown in a similar way as above that

Theorem 28. *In $\mathcal{M}_R$, if A and B are distinct maximal standard disjunctive forms of $\mathcal{L}$, then $[A]^+_r \neq [B]^+_r$.*

and infer decidability as before.

7. Completeness

Given the results established in the previous sections, the completeness proofs are all easy and follow exactly the same shape.

Theorem 29. *(**Completeness**) For all equivalences $A \approx B \in \mathcal{L}_\approx$,*

(1) if $\models_e A \approx B$ then $\vdash_e A \approx B$,
(2) if $\models_c A \approx B$ then $\vdash_c A \approx B$,
(3) if $\models_f A \approx B$ then $\vdash_f A \approx B$, and
(4) if $\models_r A \approx B$ then $\vdash_r A \approx B$.

Proof. For (1), suppose $A \approx B$ is valid under the exact interpretation of $\approx$. Then $[A]^+_e = [B]^+_e$ in $\mathcal{M}_E$. By the normal form theorems, A and B are provably equivalent within $\mathfrak{D}_e$ to full standard disjunctive forms A^* and B^*. By the soundness theorem 4, $[A^*]^+_e = [B^*]^+_e$. By the exact canonical model theorem 21, $A^* = B^*$. So A^* and B^* are provably equivalent, and thus so are A and B. The other cases are exactly analogous. □

Acknowledgements

[blinded]

References

Anglberger, A., J. Korbmacher, and F. Faroldi (2016). An Exact Truthmaker Semantics for Permission and Obligation. In O. Roy, A. Tamminga, and M. Willer (Eds.), *Deontic Logic and Normative Systems, 13th International Conference, DEON 2016*, pp. 16–31. London: College Publications.

Correia, F. (2016). On the Logic of Factual Equivalence. *The Review of Symbolic Logic 9(1)*, 103–122.

Fine, K. (2012a). Counterfactuals without possible worlds. *The Journal of Philosophy 109(3)*, 221–246.

Fine, K. (2012b). Guide to Ground. In F. Correia and B. Schnieder (Eds.), *Metaphysical Grounding: Understanding the Structure of Reality*, pp. 37–80. Cambridge: Cambridge University Press.

Fine, K. (2012c). The Pure Logic of Ground. *The Review of Symbolic Logic 5(1)*, 1–25.

Fine, K. (2016). Angellic Content. *Journal of Philosophical Logic 45(2)*, 199–226.

Fine, K. (2017a). A Theory of Truthmaker Content I: Conjunction, Disjunction and Negation. *Journal of Philosophical Logic 46(6)*, 625–674.

Fine, K. (2017b). A Theory of Truthmaker Content II: Subject Matter, Common Content, Remainder and Ground. *Journal of Philosophical Logic 46(6)*, 675–702.

Fine, K. (2017c). Truthmaker Semantics. In B. Hale, C. Wright, and A. Miller (Eds.), *A Companion to the Philosophy of Language*, pp. 556–577. John Wiley & Sons Ltd.

Fine, K. (2018a). Compliance and Command I—Categorical Imperatives. *The Review of Symbolic Logic 11(4)*, 609–633.

Fine, K. (2018b). Compliance and Command II—Imperatives and Deontics. *The Review of Symbolic Logic 11(4)*, 634–664.

Fine, K. (2019). Verisimilitude and Truthmaking. *Erkenntnis Online First*, 1–38.

Fine, K. (202x). A Theory of Partial Truth. Forthcoming in a volume for David Kaplan.

Krämer, S. (2017). A Hyperintensional Criterion of Irrelevance. *Synthese 194*, 2917–2930.

Krämer, S. (202x). The Whole Truth. Forthcoming in a volume for Kit Fine.

Krämer, S. and S. Roski (2015). A Note on the Logic of Worldly Ground. *Thought: A Journal of Philosophy 4(1)*, 59–68.