GROUND-THEORETIC EQUIVALENCE

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Abstract. Say that two sentences are ground-theoretically equivalent iff they are interchangeable salva veritate in grounding contexts. Notoriously, ground-theoretic equivalence is a hyperintensional matter: even logically equivalent sentences may fail to be interchangeable in grounding contexts. Still, there seem to be some substantive, general principles of ground-theoretic equivalence. For example, it seems plausible that any sentences of the form $A \wedge B$ and $B \wedge A$ are ground-theoretically equivalent. What, then, are in general the conditions for two sentences to stand in the relation of ground-theoretical equivalence, and what are the logical features of that relation? This paper develops and defends an answer to these questions based on the mode-ified truthmaker theory of content presented in Stephan Krämer's recent paper 'Towards a theory of ground-theoretic content' (2018).

1. Introduction

According to a widely held metaphysical picture, the world is not a mere aggregate of facts, but rather a *structured* whole, with some of its members obtaining *in virtue of* some other members. In such a case, the facts in virtue of which some other fact obtains are then said to collectively *(metaphysically) ground* the latter fact.Recently, considerable efforts have been made to try and clarify this picture by working out the general theory and logic of the grounding relations.[1] The present paper contributes to this project by addressing the question of *ground-theoretic equivalence*—roughly, the question under which conditions two sentences are interchangeable salva veritate in a grounding statement.

The question is important. First of all, it is of great significance for the general theory of ground, since one's options in developing the latter are often constrained by how one

[1] The central pioneering contributions to the study of the logic of ground are Batchelor (2010); Correia (2010); Fine (2010, 2012b,a); Rosen (2010); Schnieder (2011). More recent work includes Correia (2014, 2017, 2016); deRosset (2013, 2014); Krämer (2013); Krämer (2018); Krämer and Roski (2015); Krämer and Roski (2017); Litland (2016); Poggiolesi (2016b, 2018).

answers the former. For a particularly simple instance of this, note that given the widely accepted assumption that nothing helps grounds itself, maintaining that A and B are ground-theoretically equivalent bars one from holding that either helps ground the other. The question of ground-theoretic equivalence may also have an impact on more general issues about the nature of the grounding relation and its relata. For instance, it has been suggested that a very fine-grained view of ground-theoretic equivalence is incompatible with a conception of ground as a *worldly* phenomenon, but requires us to see ground as relating *representations* of the world rather than the world itself.[2] Beyond those connections within the theory of ground, the question of ground-theoretic equivalence may also have wider implications. For example, Kris McDaniel (2015) has argued that we should individuate propositions in general in terms of their place in the network of grounding relations. On this view, the question of ground-theoretic equivalence turns into the more general question of propositional identity.

There is a broad consensus in the current debate on certain *partial* answers to the question of ground-theoretic equivalence. In particular, most participants in that debate agree that the logical equivalence of two sentences is *not* sufficient to render them ground-theoretically equivalent. For example, if A is true, then this is taken to entail that A grounds $A \vee \neg A$, but not that A grounds $B \vee \neg B$—even though $A \vee \neg A$ and $B \vee \neg B$ are of course logically equivalent. On the other hand, most parties will also agree that there are *some* non-trivial cases of ground-theoretic equivalence. Perhaps the least contentious examples arise from the commutativity of conjunction and disjunction: pairs of the form $A \wedge B$ and $B \wedge A$, or $A \vee B$ and $B \vee A$, are presumably always ground-theoretically equivalent.

But beyond these points, there is no consensus as regards the conditions that are necessary and sufficient for ground-theoretic equivalence. Indeed, it is not so much that people disagree about what conditions are necessary and what conditions are sufficient. The problem is rather that so far, the published literature contains only very few general and appropriately developed answers to the question. As a result, there is at present no clear understanding of the range of potentially viable answers, let alone their respective merits or demerits.

My aim in this paper is to improve on this situtation in two ways. First, I shall develop and defend a novel account of ground-theoretic equivalence, based on the theory of ground-theoretic content recently put forward in Krämer (2018). To that end, I first clarify the question of ground-theoretic equivalence and lay down some desiderata on

[2] This has been urged in particular by Fabrice Correia, cf. (Correia, 2010: p. 256f, 264ff),(Correia, 2017: p. 508).

a satisfactory answer (§2). I then briefly introduce Krämer's theory of ground-theoretic content, describe the range of possible accounts of ground-theoretic equivalence that can be formulated within that theory, and provide some reasons for favouring a particular one of them (§3). Second, I shall compare the account to rival approaches and try to clarify their interrelations, as well as highlight what I take to be distinctive strengths of my own view. I have aimed to keep the discussion in the main text as informal and accessible as possible without significantly compromising on accuracy. All technical details, including soundness and completeness proofs, can be found in a formal appendix.

2. Preliminaries

To a first approximation, and in keeping with common practice in the debate, ground will here be understood as the relation of one fact obtaining in virtue of others. Paradigmatic examples include: the fact that the ball is scarlet grounds the fact that the ball is red, the fact that the ball is red and the fact that the ball is round jointly ground the fact that the ball is red and round, and the fact that the ball is round grounds the fact that the ball is round or square. When some facts jointly ground another, each fact among the former may be said to *partially* ground the latter, and we may speak of a fact's or some facts' *fully* grounding another to distinguish the primarily, full sense of ground from the partial one.

Two qualifications are in order. First, there is disagreement in the current debate concerning whether ground should be seen as relating facts or rather truths. Second, there is disagreement as to whether ground should, strictly speaking, be conceived as a relation at all. Some authors are of the view that ground is best expressed by means of a sentential connective akin to 'because' rather than by a relational predicate like 'ground(s)' and thus see no need to countenance a genuine relation of grounding.[3] For the purposes of this paper, we need not decide either issue. Informally and to facilitate presentation, I shall continue to speak in the relational mode, and describe ground as relating either facts or truths. Formally, I shall use sentential connectives symbolizing ground, for example writing $P_1, P_2, \ldots < Q$ to say that the fact that P_1, the fact that P_2, and ... fully ground the fact that Q.

Above, I introduced the question of ground-theoretic equivalence as, roughly, the question under what conditions two sentences are interchangeable salva veritate in a grounding statement. I now wish to refine this question somewhat. First of all, the

3 This view is held, for example, by Fine (e.g. 2012a) and by Correia (e.g. 2010). Rosen (2010) and Audi (2012) are among those defending the opposing view.

question I shall focus on is when sentences are interchangeable in an *argument place* of an expression of ground. The conditions under which this holds may differ from the conditions under which sentences are exchangeable when embedded within a larger sentence that occupies such an argument place. Second, the expressions of ground adverted to should be thought of as connectives like $<$, whose argument places may be filled by one or more sentences.[4] Third, I shall only be interested in cases when sentences are so interchangeable for a particular kind of reason. For suppose that two sentences A and B are interchangeable salva veritate in the argument places of $<$. Then this may be so for at least three quite different kinds of reasons, and to obtain a natural and fruitful understanding of ground-theoretic equivalence, we should abstract from two of them. First, A and B may be interchangeable simply because they are both *false*. For in that case, due to the factivity of ground, any statements of the form $\Gamma, A < C$ and $\Gamma < A$ will be false, and so will the results $\Gamma, B < C$ and $\Gamma < B$ of replacing A by B. Second, A and B may be interchangeable even though the facts they state differ with respect to their grounds, or with respect to what they ground, simply because these differences cannot be expressed in the language under consideration. Thus, it may be that the fact stated by A is fully grounded by the fact that P, and that this fact is not a full ground of the fact stated by B, but since the background language lacks a sentence stating the fact that P, A and B are nevertheless freely interchangeable in the argument places of $<$ without change of truth-value. Third, it may be that A and B are alike in every respect of meaning or content to which ground is sensitive. This seems to hold, for example, of any pair of the form $A \wedge B$ and $B \wedge A$. To the extent that these differ semantically at all, the differences they exhibit seem to be of a kind to which ground is indifferent. I shall call sentences ground-theoretically equivalent iff they are interchangeable for this third kind of reason.

Having clarified the content of the question of ground-theoretic equivalence, let us consider what might be reasonable desiderata for an adequate answer to the question. Ideally, it seems, an answer would be both fully general and maximally instructive. That is, it would specify necessary and sufficient conditions for *any* two sentences to be ground-theoretically equivalent (full generality), and the conditions it specifies would be such that we can easily ascertain with respect to any particular two sentences whether they are satisfied (maximal instructiveness). However, a single answer with both features is too much to hope for. A fully general answer would need to apply to any pair

[4] If the predicational mode of formulating statements of ground is preferred, the argument places will be filled with one or more singular terms for facts, and the canonical form of the latter would be 'the fact that P'. Our question would then have to be aimed at the interchangeability of two sentences in the position of 'P' in the canonical fact-designators.

of sentences from any language, but a maximally instructive answer can in general be expected at best for sentence-pairs from some well-defined formal language.

The next best thing would consist in two complementary answers, one achieving full generality at the cost of instructiveness, the other achieving maximal instructiveness at the cost of generality. This is analogous to what we have in the case of intensional equivalence. On the one hand, we can say with full generality that any given two sentences are intensionally equivalent just in case they are true in the same possible worlds. The notion of truth in a possible world has some intuitive content, so this is at least somewhat helpful. At the same time, it is not in general straightforward to ascertain, with respect to an arbitrary pair of sentences, whether the condition of truth in the same possible worlds is indeed satisfied.[5] The answer is therefore not maximally instructive. On the other hand, we can give a maximally instructive answer for certain special cases of the question. For instance, any standard propositional modal logic provides us with a mechanical procedure for ascertaining whether two sentences of the relevant propositional language are intensionally equivalent in virtue of their logical form (under the account of intensional equivalence for which the logic is adequate).[6]

I shall aim for an account that is in this way analogous to the account of intensional equivalence. That is, I shall demand of an adequate answer to the question of ground-theoretic equivalence that it comprise two components: one general answer, analogous to the claim that two sentences are intensionally equivalent iff true in exactly the same possible worlds, and one restricted but formally precise answer, describing a deductive system allowing us to derive exactly those statements of ground-theoretic equivalence between pairs of a formal propositional language that obtain in virtue of the logical form of the sentences involved.[7] The next section develops an answer that meets these conditions based on Krämer's mode-ified truthmaker theory of content.[8]

5 For example, it is extremely difficult to ascertain whether '2+2=4' is true in the same worlds as 'every even integer greater than 2 can be written as the sum of two primes', a sentence expressing the famously unsolved Goldbach conjecture.

6 Thanks to an anonymous referee for pressing me to elaborate on the claims made in this and the previous paragraph.

7 One might, of course, reasonably hope for a somewhat more general formal answer, perhaps one that is adequate for a richer language or which allows us to derive all the statements of ground-theoretic equivalence entailed by an arbitrary set of premises. For the purposes of this paper, however, we do better to focus on the simpler class of cases. This will simplify the formal side of things and it will facilitate comparison with previous relevant work such as Correia (2016) and Fine (2016), which also focuses on this class.

8 The label 'mode-ified' is from Krämer (2018) and motivated by the fact that Krämer crucially distinguishes different *modes* of truthmaking.

3. The Mode-ified Truthmaker Account

For a sentence to be ground-theoretically equivalent to another is for the two to be semantically alike in every respect to which ground is sensitive. But to what semantic features of a sentence is ground sensitive? On the view developed in Krämer (2018), the answer is that ground is sensitive to (a) what states verify a sentence and (b) *how*, or in what *modes*, they do so. §3.1 clarifies how the key notions of verification, states, and modes are understood within mode-ified truthmaker theory. On that basis, §3.2 then develops an answer to the general question of ground-theoretic equivalence, before §3.3 answers the formal question.

3.1. **Mode-ified Truthmaking.** As its name suggests, the mode-ified truthmaker theory of content is a modification of the truthmaker theory of content developed by Kit Fine (see e.g. his 2016; 2017a; 2017b). Most of the details of the truthmaker account are not essential for understanding the mode-fied version. It suffices to note the following basic points. The key concept of the approach is that of a proposition being (exactly[9]) verified or falsified by a state. A state is conceived as a part, or fragment, of a world, distinguished from the latter by the fact that it need not be complete, but may leave open the truth-value of many propositions. The space of states comes equipped with a relation of part-whole, and given any set of states $T = \{s_1, s_2, \ldots\}$ we may form the fusion $\bigsqcup T = s_1 \sqcup s_2 \sqcup \ldots$ of its members, which is the smallest state of which they are all part. The operation of fusion is crucial to the treatment of conjunction: the states verifying a conjunction are exactly those states that may be obtained by fusing a verifier of one conjunct with a verifier of the other conjunct. A disjunction, by contrast, is verified by the states verifying either disjunct. Dually, a disjunction is falsified by any fusions of falsifiers of the disjuncts, and a conjunction is falsified by the falsifiers of the conjuncts.

The notion of verification (and analogously falsification) incorporates a strong requirement of *relevance*. In order for a state to verify a proposition, it must be wholly relevant to the truth of the proposition. Thus, the state of two plus two being four does not verify the proposition that grass is green or not: even though the state's obtaining necessitates the truth of the proposition, it is irrelevant to it. Moreover, the state of it being sunny and warm does not verify the proposition that it is sunny, since it is not *wholly* relevant to the latter: it contains as an irrelevant part the state of it being warm.

We turn finally to *modes* of verification. The basic observation motivating the mode-ified truthmaker approach is this. Intuitively, for some propositions, there are several

9 Fine distinguishes several notions of truthmaking of which the exact notion is the narrowest or most demanding one. The other notions are of no special importance for our purposes. When speaking of truthmakers or verification, I shall henceforth always mean the exact variety.

ways, or *modes*, in which a state might verify the proposition (analogous remarks apply with respect to falsification). Disjunctive propositions are perhaps the most compelling example. Thus, consider the proposition that this ball is red or blue. This proposition might be made true, for example, by the state of the ball being scarlet—call this state s—, or by the state of the ball being navy blue—call this state n. Now note that the state s also verifies the proposition that this ball is red, and it seems plausible to say that s verifies the proposition that this ball is red or blue *by* verifying the proposition that this ball is red. Similarly, the state n also verifies the proposition that this ball is blue, and it seems plausible to say that n verifies the proposition that this ball is red or blue *by* verifying the proposition that this ball is blue. In this way, we may distinguish between at least two different modes of verifying the proposition that this ball is red or blue: the mode of verifying it by verifying the one disjunct, that this ball is red, and the mode of verifying it by verifying the other disjunct, that this ball is blue.[10]

The key features of modes of verification, in bullet point fashion, are these:

(1) Modes are subject to a fullness or sufficiency condition. So verifying a proposition P is a mode of verifying a proposition Q only if verifying P is *sufficient* for verifying Q.[11]
(2) Some propositions may be verified by verifying not a single proposition P, but a *multitude* of propositions $P_1, P_2, \ldots$ The most compelling example is that of a conjunction $P \wedge Q$ which may be verified by verifying both P and Q.
(3) Modes can be combined. For instance, if P may be verified by verifying $P_1, P_2, \ldots$ and Q may be verified by verifying $Q_1, Q_2, \ldots$ then plausibly, $P \wedge Q$ may be verified in a combination of these two modes, by verifying $P_1, P_2, \ldots, Q_1, Q_2, \ldots$
(4) There may be propositions P for which there are no propositions $P_1, P_2, \ldots$ such that P may be verified by verifying $P_1, P_2, \ldots$ P is then said to be verified in a *direct*, unmediated way by any verifying states.
(5) Talk of verification, and modes of verification, is to be understood *non-factively*: that verifying P is a mode of verifying Q does not imply that P or Q are true. Indeed, in the intended sense, it does not even imply that P or Q are *possibly*

[10] In order to further clarify the notion of a mode of verification, one might consider analysing it in terms of ground: prima facie, one might think that for a state s to verify P by verifying $Q, R, \ldots$ is for s's verifying $Q, R, \ldots$ to ground s's verifying P. But this is not the place to further pursue this idea.

[11] This bears highlighting since in ordinary language, 'by' tolerates partial ways of doing something: by saying that someone got into the house by breaking a window we do not imply that breaking the window was on its own sufficient for the person to get into the house.

true. For example, verifying $P, \neg P$ is a mode of verifying $P \wedge \neg P$, even though it is logically impossible for $P \wedge \neg P$ to be true.[12]

(6) The relation of verification in a mode has certain important structural properties similar to those standardly associated with grounding. For example, like grounding, it is non-monotonic. Thus, it may happen that a proposition can be verified by verifying P, but not by verifying P, Q. Further structural properties that mode-ified verification plausibly exhibits include irreflexivity—no proposition is verified by verifying itself—and transitivity—if P is verified by verifying Q, and Q by verifying R, then P is verified by verifying R.[13]

For ease of expression, we informally refer to the mode of verifying a proposition by verifying $P_1, P_2, \ldots$ as the mode via-$P_1, P_2, \ldots$ Note that to each mode m = via-$P_1, P_2, \ldots$ there corresponds the set $|m|$ of propositions $\{P_1, P_2, \ldots\}$. It is by reference to this set of propositions that relationships of *ground* may ultimately be defined.

First, though, we need to say how the notion of a proposition is understood within the mode-ified truthmaker account. Any non-empty set of modes is a *unilateral* proposition. The presence of a mode m in a unilateral proposition P represents that m is a mode of verifying P. More precisely, if m = via-$P_1, P_2, \ldots$, then whenever $s = s_1 \sqcup s_2 \sqcup \ldots$ with each s_i verifying the corresponding P_i, s verifies P by verifying $P_1, P_2, \ldots$ If m is the mode of being directly verified by state s, then the presence of m in a proposition P represents that P is directly verified by s.

Any pair of unilateral propositions is a *bilateral* proposition. While unilateral propositions are referred to by uppercase letters $P, Q, \ldots$, bilateral propositions are referred to by bold-face versions of these. The first (second) component of a bilateral proposition $\mathbf{P}$ is called its positive (negative) content and denoted $\mathbf{P}^+$ ($\mathbf{P}^-$). That a mode m is a member of $\mathbf{P}^+$ ($\mathbf{P}^-$) represents that m is a mode of verifying (falsifying) $\mathbf{P}$.

Finally, relations of full, partial, weak, and strict ground for unilateral contents are defined as follows:

Γ strictly fully grounds P ($\Gamma < P$)	$:\leftrightarrow \Gamma = \|m\|$ for some mode $m \in P$
Q strictly partially grounds P ($Q \prec P$)	$:\leftrightarrow Q \in \Gamma$ for some $\Gamma < P$
Γ weakly full grounds P ($\Gamma \leq P$)	$:\leftrightarrow \Gamma = \{P\}$ or $\Gamma < P$ or $\Gamma \backslash \{P\} < P$
Q weakly partially grounds P ($Q \preceq P$)	$:\leftrightarrow Q = P$ or $Q \prec P$

[12] Similarly, when mention is made of states (of affairs) verifying certain propositions, the pertinent notion of a state should be taken to encompass non-actual and even impossible states as well as actual ones.

[13] Under the ground-theoretic analysis of modes of verification suggested in footnote 10, we might be able to derive that modes of verification have these properties from the corresponding ground-theoretic assumptions.

Grounding between bilateral propositions holds just in case it obtains between the positive components. Note that these definitions target a *non-factive* understanding of ground on which falsities may ground each other just as much as truths may. The more familiar factive notion can easily be defined in terms of the non-factive one by adding the condition that grounds and groundee be true.[14] For present purposes, however, there is no need to do so.

Krämer (2018) then shows that under a natural account of conjunction, disjunction, and negation, grounding interacts with these truth-functions in accordance with some very attractive and intuitively compelling principles.[15] To state them succinctly, let $\mathbf{\Gamma} \leq \{\mathbf{P}_1, \mathbf{P}_2, \ldots\}$ abbreviate that for some sets $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \ldots$ with $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_2 \cup \ldots$, $\mathbf{\Gamma}_1 \leq \mathbf{P}_1$ and $\mathbf{\Gamma}_2 \leq \mathbf{P}_2$ and ... Then the central result is the following set of equivalences (theorems 5 and 6 of the appendix in Krämer (2018)):

(<∧): $\mathbf{\Gamma} < \mathbf{P} \wedge \mathbf{Q}$ iff $\mathbf{\Gamma} \leq \{\mathbf{P}, \mathbf{Q}\}$
(<∨): $\mathbf{\Gamma} < \mathbf{P} \vee \mathbf{Q}$ iff $\mathbf{\Gamma} \leq \mathbf{P}$ or $\mathbf{\Gamma} \leq \mathbf{Q}$ or $\mathbf{\Gamma} \leq \{\mathbf{P}, \mathbf{Q}\}$
(<¬¬): $\mathbf{\Gamma} < \neg\neg\mathbf{P}$ iff $\mathbf{\Gamma} \leq \mathbf{P}$
(<¬∧): $\mathbf{\Gamma} < \neg(\mathbf{P} \wedge \mathbf{Q})$ iff $\mathbf{\Gamma} \leq \neg\mathbf{P}$ or $\mathbf{\Gamma} \leq \neg\mathbf{Q}$ or $\mathbf{\Gamma} \leq \{\neg\mathbf{P}, \neg\mathbf{Q}\}$
(<¬∨): $\mathbf{\Gamma} < \neg(\mathbf{P} \vee \mathbf{Q})$ iff $\mathbf{\Gamma} \leq \{\neg\mathbf{P}, \neg\mathbf{Q}\}$

I shall sometimes refer to these principles as introduction (elimination) principles for the relevant connectives, when only the right-to-left (left-to-right) direction of the bi-conditional is at issue.

3.2. **The General Account of Ground-Theoretic Equivalence.** The general Mode-ified Truthmaker Account of ground-theoretic equivalence may be stated as follows:[16]

MTA: Sentences S and T are ground-theoretically equivalent if and only if S and T are verified by the same states in the same modes.

Given the account of the previous section, this simplifies to the claim that sentences are ground-theoretically equivalent iff the positive components of the bilateral propositions

[14] Cf. (Krämer, 2018: pp. 795f, 798). On the distinction between factive and non-factive grounding, see also (Fine, 2012a: p. 48ff) and (Correia, 2014: p. 36).—To define a notion of truth in the present framework, some modes may be designated as *actual*, i.e. such that some state verifies in the relevant mode, and a proposition may then be counted as true just in case it is verified in some actual mode.

[15] These principles, in a slightly different form, were first proposed by (Fine, 2012a: sections 7 and 8). Versions of in particular the left-to-right directions of the bi-conditionals are endorsed in numerous works on the logic of ground, including Batchelor (2010); Rosen (2010); Schnieder (2011); Correia (2014, 2017).

[16] Here and in what follows, talk of sentences being verified (in a certain mode) is to be read as shorthand for talk of the sentence's positive content being verified (in the relevant mode).

expressed by the sentences are identical. Writing $[S]$ for the proposition expressed by S:

(A) Sentences S and T are ground-theoretically equivalent iff $[S]^+ = [T]^+$

I shall make the widely accepted assumption that no proposition is a strict full ground of itself. As Krämer shows (2018: p. 805f), given that assumption, the condition of identity of positive content is equivalent to what we may call sameness of overall ground-theoretic profile. More precisely, let us write $\approx$ for the relation defined as follows:

: $\mathbf{P} \approx \mathbf{Q}$ iff

(i) for all $\mathbf{\Gamma}$: $\mathbf{\Gamma} < \mathbf{P}$ iff $\mathbf{\Gamma} < \mathbf{Q}$, and

(ii) for all $\mathbf{\Delta}$ and $\mathbf{R}$: $\mathbf{\Delta}, \mathbf{P} < \mathbf{R}$ iff $\mathbf{\Delta}, \mathbf{Q} < \mathbf{R}$.

Thus, $\mathbf{P} \approx \mathbf{Q}$ iff $\mathbf{P}$ and $\mathbf{Q}$ participate in exactly the same relationships of non-factive strict full ground.[17] Since all the other relationships of ground are defined in terms of non-factive strict full ground, $\mathbf{P} \approx \mathbf{Q}$ implies overall sameness of ground-theoretic profile. Then assuming the irreflexivity of ground,

(B) $\mathbf{P}^+ = \mathbf{Q}^+$ iff $\mathbf{P} \approx \mathbf{Q}$

I shall henceforth use ‘ground-theoretic equivalence’ both for the relationship between sentences characterized in the beginning of the paper and the relation $\approx$ between propositions, and I shall often tacitly rely on the equivalences (A) and (B). For convenience, I mostly carry out the subsequent investigation purely on the level of content, avoiding the distraction of a detour through language whenever possible.

Now in order to determine whether two given propositions are ground-theoretically equivalent, we may need to decide whether a certain mode m = via-$P_1, P_2, \ldots$ is *the same mode* as a mode n = via-$Q_1, Q_2, \ldots$ For instance, suppose we wish to know whether $\mathbf{P} \wedge \mathbf{Q} \approx \mathbf{Q} \wedge \mathbf{P}$. Under a natural account of conjunction, this will be so just in case the mode via-$\mathbf{P}^+, \mathbf{Q}^+$ is identical to the mode via-$\mathbf{Q}^+, \mathbf{P}^+$.

[17] An anonymous referee has raised the question whether condition (ii) is needed in the definition of $\approx$. Condition (ii) is redundant just in case whenever two propositions have the same grounds, they ground the same things. This is not in general so under the mode-theoretic approach, it depends on just how modes are individuated—a matter we shall take up momentarily. We shall see then that under the most coarse-grained of the available views, condition (ii) is indeed redundant, but not on the others. As it happens, however, condition (i) *is* in general redundant within the mode-theoretic framework when irreflexivity of $<$ is assumed. For suppose $\mathbf{P}$ and $\mathbf{Q}$ ground the same things. By $(< \neg\neg)$, $\mathbf{Q} < \neg\neg\mathbf{Q}$, and so $\mathbf{P} < \neg\neg\mathbf{Q}$, which entails again by $(< \neg\neg)$ that $\mathbf{P} \leq \mathbf{Q}$. By parallel reasoning, $\mathbf{Q} \leq \mathbf{P}$. but given irreflexivity, it is easy to verify that mutual weak full ground implies sameness of positive content and thereby also sameness of grounds. I have nevertheless included condition (i) since this is how Krämer defines the relation, and since this makes the conceptual connection to ground-theoretic equivalence as introduced in the beginning of the paper much more explicit.

It is clear, first of all, that the modes via-$P_1, P_2, \ldots$ and via-$Q_1, Q_2, \ldots$ are identical if the *sequences* $\langle P_1, P_2, \ldots\rangle$ and $\langle Q_1, Q_2, \ldots\rangle$ are identical. We may assume, furthermore, that the modes via-$P_1, P_2, \ldots$ and via-$Q_1, Q_2, \ldots$ are identical *only if* the *sets* $\{P_1, P_2, \ldots\}$ and $\{Q_1, Q_2, \ldots\}$ are identical.[18] In light of this, there are two natural further principles concerning the individuation of modes to consider. The first is that modes are *order-insensitive*: the modes via-$P_1, P_2, \ldots$ and via-$Q_1, Q_2, \ldots$ are identical whenever the sequences $\langle P_1, P_2, \ldots\rangle$ and $\langle Q_1, Q_2, \ldots\rangle$ differ at most by the order of the propositions they contain, i.e. when the *multi-sets* $\lfloor P_1, P_2, \ldots\rfloor$ and $\lfloor Q_1, Q_2, \ldots\rfloor$ are the same.[19] The second is that modes are order- and *repetition-insensitive*: the modes via-$P_1, P_2, \ldots$ and via-$Q_1, Q_2, \ldots$ are identical whenever the sequences $\langle P_1, P_2, \ldots\rangle$ and $\langle Q_1, Q_2, \ldots\rangle$ differ at most by the order or the number of occurrences of the propositions they contain, i.e. when the *sets* $\{P_1, P_2, \ldots\}$ and $\{Q_1, Q_2, \ldots\}$ are the same.

I shall defend the view that our conception of modes should be order-insensitive but repetition-sensitive. §3.2.1 makes the case for order-insensitivity, and §3.2.2 argues for repetition-sensitivity. §3.2.3 develops and responds to a possible objection.

3.2.1. *Against Order-Sensitivity.* For the purposes of this discussion, I shall assume that if an order-sensitive conception of modes is assumed, then $\mathbf{P} \wedge \mathbf{Q}$ will not in general be verified via $\mathbf{Q}^+, \mathbf{P}^+$, although it will be verified via $\mathbf{P}^+, \mathbf{Q}^+$. This seems reasonable. First, the assumption holds under the account of conjunction in Krämer (2018). Second, although it would be possible to adjust that account of conjunction so as to avoid this result, it is hard to see what point the distinction between the modes via-$\mathbf{P}^+, \mathbf{Q}^+$ and via-$\mathbf{Q}^+, \mathbf{P}^+$ could have if not to distinguish between the conjunctions $\mathbf{P} \wedge \mathbf{Q}$ and $\mathbf{Q} \wedge \mathbf{P}$.

The first point in favour of order-insensitivity is that an order-sensitive conception of modes has counter-intuitive consequences, such as that $\mathbf{P} \wedge \mathbf{Q}$ grounds $\neg\neg(\mathbf{P} \wedge \mathbf{Q})$ whereas $\mathbf{Q} \wedge \mathbf{P}$ does not.[20] In addition, we can defend order-insensitivity by appeal to the way that modes of verification have informally been introduced. Suppose that Q and R are distinct unilateral propositions, and that via-Q, R and via-R, Q are different

18 Indeed, we have assumed as much when we claimed that to each mode m there corresponds a unique set $|m|$ of propositions. The assumption is crucial for the derivation of the main results in the logic of ground of Krämer (2018); see esp. the appendix of that paper.

19 A multi-set is like a set except in that it may contain the same item more than once. I write $\lfloor P, Q \ldots\rfloor$ for the multi-set including exactly $P, Q, \ldots$, each exactly as many times as it is listed.

20 Suppose that via-$\mathbf{P}^+, \mathbf{Q}^+$ and via-$\mathbf{Q}^+, \mathbf{P}^+$ are distinct modes, so $\mathbf{P} \wedge \mathbf{Q}$ and $\mathbf{Q} \wedge \mathbf{P}$ are ground-theoretically inequivalent. Then $\mathbf{P} \wedge \mathbf{Q} < \neg\neg(\mathbf{P} \wedge \mathbf{Q})$. But suppose for reductio that also $\mathbf{Q} \wedge \mathbf{P} < \neg\neg(\mathbf{P} \wedge \mathbf{Q})$. By the elimination principle for double negation, it follows that $\mathbf{Q} \wedge \mathbf{P} \leq \mathbf{P} \wedge \mathbf{Q}$. Since $\mathbf{P} \wedge \mathbf{Q} \not\approx \mathbf{Q} \wedge \mathbf{P}$, it follows that $\mathbf{Q} \wedge \mathbf{P}$ is *strict* full ground of $\mathbf{P} \wedge \mathbf{Q}$, which is absurd. So $\mathbf{Q} \wedge \mathbf{P} \not< \neg\neg(\mathbf{P} \wedge \mathbf{Q})$.

modes. We are then making a distinction between the mode of verifying a proposition by verifying Q, R and the mode of verifying the proposition by verifying R, Q. This is plausible only to the extent that some difference can be made out between a state's verifying Q, R and the same state's verifying R, Q.

The relevant notion of verifying a sequence of propositions is explicitly defined by Krämer in terms of the notion of verifying a single proposition in the following way: 'the propositions $P_1, P_2, \ldots$ are verified by a [state] iff the [state] is the fusion of some [states] $s_1, s_2, \ldots$ verifying $P_1, P_2, \ldots$ respectively' (2018: 792).[21] Under this definition, then, to say that a state s verifies a given proposition P via-Q, R is in effect to say that s verifies P by being the fusion of some states t, u verifying Q, R, respectively. And to say, in constrast, that state s verifies P via R, Q is to say that s verifies P by being the fusion of some states t, u verifying R, Q, respectively. Given that the operation of fusion is an operation on sets of states and thus insensitive to order, it would seem wholly implausible to take these two statements to describe two distinct ways in which s verifies P. Moreover, it is hard to see how any plausible alternative definition of the verification of a sequence of propositions could lead to a different assessment.

3.2.2. *For Repetition-Sensitivity.* Similarly as in the previous case of order-sensitivity, for the purposes of our discussion of repetition-sensitivity, we shall make an assumption about how the matter manifests itself in the theory of ground-theoretic equivalence. Specifically, we shall assume that if a repetition-sensitive conception of modes is adopted, then the propositions $\mathbf{P} \wedge \mathbf{P}$, $\mathbf{P} \vee \mathbf{P}$, and $\neg\neg\mathbf{P}$ will be pairwise inequivalent, whereas they are pairwise equivalent under a repetition-insensitive conception. Again, this seems reasonable. First, the assumption holds under the account of the truth-functional operations in Krämer (2018): $\mathbf{P} \wedge \mathbf{P}$ is then verified via $\mathbf{P}^+, \mathbf{P}^+$ but not via $\mathbf{P}^+$, the opposite is true of $\neg\neg\mathbf{P}$, while $\mathbf{P} \vee \mathbf{P}$ is verified both via $\mathbf{P}^+$ and via $\mathbf{P}^+, \mathbf{P}^+$. Second, although it might be possible to adjust the definitions so as to avoid this result, it is hard to see what point the distinction between modes differing only with respect to repetition could retain under such an alternative set of definitions.[22] Third, as we

21 Krämer here speaks of facts—i.e. states that actually obtain—rather than states, but given how his view is later developed, the appropriate general definition must appeal to states (cf. (Krämer, 2018: p. 794f)).

22 It might perhaps be argued that $\mathbf{P} \vee \mathbf{P}$ should not be taken to be verified via $\mathbf{P}^+, \mathbf{P}^+$, but only via $\mathbf{P}^+$. It would then turn out ground-theoretically equivalent to $\neg\neg\mathbf{P}$ even under a repetition-sensitive conception of modes. A version of the central point above would then still remain, however. For the purposes of our arguments below, we only need the assumption that $\mathbf{P} \wedge \mathbf{P}$ is ground-theoretically distinguished from both $\mathbf{P} \vee \mathbf{P}$ and $\neg\neg\mathbf{P}$ just in case a repetition-sensitive conception is adopted. And

shall see, the ability to render the above pairs inequivalent is precisely what makes the repetition-sensitive conception seem preferable.

We note first that a repetition-sensitive conception of modes seems to yield a more intuitive account than a repetition-insensitive one. For consider the truth that snow is black or (snow is white and snow is white). It obtains because (snow is white and snow is white). From the ground-theoretic equivalence between the proposition that (snow is white and snow is white) and the proposition that (it is not the case that snow is not white), we could infer that the truth that snow is black or (snow is white and snow is white) holds because it is not the case that snow is not white. This seems a rather odd thing to say.[23] So the relevant intuitions about the plausibility of 'because'-statements seem to speak against imposing the requirement of repetition-insensitivity.

In addition, we can defend repetition-sensitivity based on the informal conception of a mode of verification. A conception of modes that is repetition-sensitive embodies a distinction between pairs of modes like via-P and via-P, P. This distinction is justified just in case there is a relevant *difference* between a state's verifying P and a state's verifying P, P. I shall now argue that there is such a difference.

Similarly as before, we proceed by applying the official definition of the notion of a state's verifying a list of proposition. We then obtain that for a state s to verify P, P is for s to be the fusion of some states t, u such that t verifies P and u verifies P. At first glance, this condition seems importantly different from the simple condition of s being a verifier of P: being the fusion of items with a certain property is not the same as being an item with that property.

One might try to object that the case of the property of verifying a proposition P is special in this regard. In particular, one can argue that the conditions in question are equivalent. Clearly, if s is a verifier of P, then since $s = s \sqcup s$, s is also the fusion of a verifier of P with a verifier of P. What about the converse? In Fine's truthmaker semantics for ground, it is assumed that whenever a state s is a fusion of states t, u both verifying P, s itself also verifies P. In that context, this assumption corresponds to the ground-theoretic principle of *amalgamation*, which says that if $\Gamma < P$ and $\Delta < P$ then $\Gamma \cup \Delta < P$. An analogous condition on modes, playing the same role with respect to the logic of ground, may also be imposed on Krämer's account (cf. (Krämer, 2018:

it would be hard to see what point the distinction between modes via-P, P and via-P could retain if not to allow for this distinction.

[23] The intuition may be even more forceful when one discusses the case schematically—even if intuitions about these sorts of statements are perhaps not in any strong sense *pre-theoretical* intuitions. If it is true that A, then it seems plausible that B or (A and A) because A and A, but it seems considerably stranger to suggest that B or (A and A) because it is not the case that not-A.

p. 804f)). So by assuming the principle of amalgamation, one might indeed argue that any fusion of verifiers of P must itself be a verifier of P.[24]

But even granting that the conditions are satisfied by the same states, this is consistent with the claim that there are important differences between them. In particular, as I shall now argue, the conditions differ with respect to what propositions a state may verify *by* satisfying them. For let s be a state verifying a proposition P, and consider the conjunction $P \wedge P$. Under the truthmaker approach, mode-ified or otherwise, the set of states verifying a conjunction $P \wedge Q$ is defined as the set of states that can be obtained by taking the fusion of a verifier of P and a verifier of Q. By that definition, what it takes for a state to verify $P \wedge P$ is for it to be the fusion of a verifier of P and a verifier of P. The state s satisfies this condition, and therefore verifies $P \wedge P$. Moreover, so the basic insight underlying the mode-ified approach, s verifies $P \wedge P$ *by* being the fusion of a verifier of P and a verifier of P. By the definition of verification of a list of propositions, we may infer that s verifies $P \wedge P$ *by* verifying P, P.

Consider now the double negation $\neg\neg P$.[25] The set of states verifying $\neg\neg P$ is defined as the set of states verifying P. Since s verifies P, it also verifies $\neg\neg P$, and under the account defended in Krämer (2018), it does so *by* verifying P. However, if no distinction is made between verifying P and verifying P, P, we may infer from this that s verifies $\neg\neg P$ by verifying P, P, i.e. by being the fusion of a verifier of P and a verifier of P—just like in the case of $P \wedge P$. But this seems implausible. Although s does satisfy this condition, it is not *in virtue of* satisfying *this* condition that s qualifies as a verifier of $\neg\neg P$. If we look at what makes it the case that a state verifies $\neg\neg P$, fusions simply don't come into it. So we have a good reason to distinguish between the modes via-P and via-P, P: $\neg\neg P$ should include the former but not the latter.

3.2.3. *Objection: Individuation from Below.* Given the usual conception of grounding as relating a plurality or set of propositions to a single proposition, adopting a repetition-sensitive conception of modes creates a kind of mismatch in the individuation of modes on the one hand and grounds on the other. The strongest argument against the repetition-sensitive conception of modes, it seems to me, holds that this mismatch

[24] It should be mentioned, though, that the principle of amalgamation is among the more contentious principles in the logic of ground advocated in Fine (2012b), and that Fine himself elsewhere professes some uneasiness about it; cf. his 2012a: 59n16.

[25] Strictly speaking, under the present approach, negation is only defined on bilateral contents. However, if P is the positive content of $\mathbf{P}$, the positive content of $\neg\neg\mathbf{P}$ depends only on P—it is the result of applying the so-called *raising* operation to P. As a harmless simplification, I here speak of that unilateral content as the double negation of P.

is undesirable. The aim of this section is to sketch such an argument, and to say how repetition-sensitivity may be defended against the objection.

Under a a repetition-insensitive view of modes, the indirect modes in which a given proposition may be verified correspond one-to-one to non-factive strict full grounds of the proposition. As a result, given any propositions **P** and **Q** that are not fundamental—i.e. they have at least one non-factive strict full ground—**P** is ground-theoretically equivalent to **Q** just in case **P** and **Q** have exactly the same grounds. So for non-fundamental propositions, sameness of grounds implies sameness of groundees. In this sense, the items in the ground-theoretic hierarchy then obey a principle of *Individuation from Below*: the identity of any given item in the hierarchy is fully determined by the part of the hierarchy below it.

I can think of two reasons to be attracted to the principle of Individuation from Below, one logical and one metaphysical. The logical reason is that it opens up the possibility of an alternative, and potentially particularly neat and elegant axiomatization of the logic of ground-theoretic equivalence, given a suitable background logic of ground. Essentially, we could utilize familiar techniques from standard quantificational logic to allow the inference to $A \leq B$ whenever $\Gamma < B$ has been derived from $\Gamma < A$ for *arbitrary* Γ. We would then only need to add a rule allowing us to infer $A \approx B$ given both $A \leq B$ and $B \leq A$.[26]

Although this is an intriguing idea, more work would need to be done before it could justify an assessment of repetition-insensitivity as generating the overall more attractive logic of ground and ground-theoretic equivalence. And even if such an assessment could be justified, there remains the question of how this fact should be weighed up against the intuitive benefits offered by the repetition-sensitive view. So far, then, the logical considerations in favour of Individuation from Below do not constitute a strong case against the repetition-sensitive conception.

The metaphysical reason to find the principle of Individuation from Below attractive is that it yields a purer, more self-contained picture of the ground-theoretic hierarchy, similar in respect of individuation to the set-theoretic hierarchy. In the latter case, the identity of each item in the hierarchy is fully accounted for in terms of its set-theoretic relationship to items that are below it. In the same way, on the repetition-insensitive view, the identity of each item in the ground-theoretic hierarchy is fully accounted for

[26] For a simple illustration of the idea, assume for arbitrary Γ that $\Gamma < A \wedge B$. Using the elimination rule for conjunction, we infer $\Gamma \leq \{A, B\}$. But since $\{A, B\} = \{B, A\}$, this is just $\Gamma \leq \{B, A\}$. Using an obvious generalization of the usual introduction rule for conjunction, we infer $\Gamma < B \wedge A$. Since Γ was arbitrary, by the rule sketched above, we may then infer $A \wedge B \leq B \wedge A$. Parallel reasoning establishes $B \wedge A \leq A \wedge B$, whence we may infer $A \wedge B \approx B \wedge A$.

in terms of its ground-theoretic relationship to items below it. Under the repetition-sensitive conception, in contrast, this is not so. Although the identity of an element in the grounding hierarchy may still be determined without reference to items *above* it, we now have to appeal to something *extraneous* to the ground-theoretic hierarchy, namely *multi-sets* of elements below the given element. In that sense, we are introducing distinctions into the ground-theoretic hierarchy with no purely ground-theoretic basis, and it might be thought that this should be avoided.

Three things may be said in response. The first is that the argument relies on a contentious 'purity' assumption concerning legitimate ways to individuate ground-theoretic content, for which a sustained defence has yet to be given. The second is to challenge the claim that the relevant distinctions have no purely ground-theoretic basis. For it seems that one natural way to understand the argument of the previous section is precisely as identifying such a basis. We argued there, recall, that $P \wedge P$ may be verified by verifying P, P, whereas $\neg\neg P$ may not be so verified. It is not implausible to regard these claims as equivalent to corresponding grounding claims: that s verifies P, P grounds that s verifies $P \wedge P$, but does not ground that s verifies $\neg\neg P$. A third response might be to accept Individuation from Below, but to insist that rather than adopt a repetition-insensitive conception of modes, we should adopt a repetition-sensitive conception of grounds. I myself have some sympathy for this suggestion, but I lack the space to further pursue it here.[27]

3.3. **The Formal Account of Ground-Theoretic Equivalence.** This section answers the formal question of ground-theoretic equivalence. Since the case made in the previous section for repetition-sensitivity is not fully decisive, and since we found the repetition-insensitive conception to carry some theoretical interest, I shall do this for both the repetition-sensitive and the repetition-insensitive conception of modes. The formal question I posed, recall, is under what conditions two sentences are ground-theoretically equivalent in virtue of their propositional logical form. To answer this question, we shall make use of a formal language $\mathcal{L}_{\approx}$ in which statements of ground-theoretic equivalence can be formulated. In our choice of a language, we shall follow the example of Correia (2016) and use a standard propositional language with connectives $\wedge$, $\vee$, and $\neg$, augmented by all expressions of the form $A \approx B$ where A, B are sentences of the propositional language (and thus do not already include occurrences of $\approx$).

[27] Note that Poggiolesi (2016b, 2018) also works with a conception of grounds as multi-sets.

The following system of axioms and rules for $\mathcal{L}_{\approx}$—I shall call it $\mathfrak{I}$ or *the intermediate system*—is sound and complete relative to the order-insensitive but repetition-sensitive conception of modes.[28,29]

(Commutativity $\vee$)	$A \vee B \approx B \vee A$
(Commutativity $\wedge$)	$A \wedge B \approx B \wedge A$
(De Morgan 1)	$\neg(A \vee B) \approx \neg A \wedge \neg B$
(De Morgan 2)	$\neg(A \wedge B) \approx \neg A \vee \neg B$
(Reflexivity)	$A \approx A$
(Symmetry)	$A \approx B \ / \ B \approx A$
(Transitivity)	$A \approx B,\ B \approx C \ / \ A \approx C$
(Preservation $\vee$)	$A \approx B \ / \ A \vee C \approx B \vee C$
(Preservation $\wedge$)	$A \approx B \ / \ A \wedge C \approx B \wedge C$
(Preservation $\neg\neg$)	$A \approx B \ / \ \neg\neg A \approx \neg\neg B$

Under a repetition-insensitive conception of modes, corresponding self-disjunctions, self-conjunctions, and double negations are ground-theoretically equivalent. As a result, this conception leads to the validation of the following additional axioms:

(Collapse $\wedge/\vee$)	$A \wedge A \approx A \vee A$
(Collapse $\vee/\neg\neg$)	$A \vee A \approx \neg\neg A$

More strikingly, under a repetition-insensitive conception, the condition of a proposition **P** being a weak full ground of a proposition **Q** is equivalent to the condition that $\mathbf{P} \vee \mathbf{Q} \approx \mathbf{Q} \vee \mathbf{Q}$ (cf. (Krämer, 2018: p. 807)). Therefore, letting $A \leq B$ abbreviate $A \vee B \approx B \vee B$, we can then state within $\mathcal{L}_{\approx}$ a form of introduction rules for $\leq$:

(Introduction $\leq\wedge$)	$A \leq B,\ A \leq C \ / \ A \leq B \wedge C$
(Introduction $\leq\vee$)	$A \leq B \ / \ A \leq B \vee C$

Adding these four principles to $\mathfrak{I}$ results in a system I shall call $\mathfrak{E}$, or *the extensional system*.[30] It is sound and complete relative to the order- and repetition-insensitive conception of modes.

28 The principles are to be read as follows. The first five principles are axiom schemata, so any instance obtained by systematically replacing the sentence letters A, B, C by propositional formulae is an axiom. The last five principles are (schematic) rules, starting with a comma-separated list of the premises, separated by a forward slash from the conclusion.

29 The proof of this result and the next is given in appendix A. The motivation for the label 'intermediate' is that relative to the range of possible standards of individuation in the mode-theoretic framework, $\mathfrak{I}$ occupies an intermediate position.

30 The motivation for the label is that given the principle of Individuation from Below, non-fundamental ground-theoretic contents are extensional with respect to their grounds in the sense that they are ground-theoretically equivalent if they have the same grounds.

4. Comparison to other Approaches

So far, there is effectively only one other approach that offers both a general account of ground-theoretic equivalence and an answer to the formal question: the (unmodeified) truthmaker account, which is developed in slightly different versions in Fine (2016) and Correia (2016). Beyond that, three partial accounts have been proposed. Correia (2017) develops, among other things, a formal account of ground-theoretic equivalence, but no informative general account that could serve to motivate the formal account. Correia (2018), in contrast, formulates an alternative account of ground, and ground-theoretic equivalence, but only partially specifies the resulting propositional logic of ground-theoretic equivalence. Poggiolesi (2016b, 2018) develops a further and interestingly different account of logical grounding and ground-theoretic equivalence. The purpose of this section is to identify both similarities and dissimilarities between these other approaches and the one defended in this paper, and to point out where I think the latter holds an advantage.

4.1. **Fine's and Correia's Truthmaker Accounts.** The *truthmaker* account may be stated as follows:

> **(TM):** Sentence S is ground-theoretically equivalent to sentence T iff S and T are exactly *verified* by exactly the same states.

This account of ground-theoretic equivalence is implicit in the truthmaker account of ground presented in (Fine, 2017b: §6). Here ground is defined as relating contents identified with sets of exact truthmakers. Sentences accordingly will turn out ground-theoretically equivalent just in case they have the same exact truthmakers.[31]

A version of this view is given a more explicit and detailed exposition in Correia (2016). Here, Correia discusses a relation he calls *factual equivalence*, which he defines as obtaining between two sentences just in case they describe the same situations. The notion of a sentence's describing a situation is proposed by Correia as a more concrete specification of Fine's notion of a sentence being verified by a state.[32] In the final

[31] Versions of the same view are also implicit in the other contributions of Fine's in which he formulates a truthmaker semantics for ground, specifically Fine (2012b,a). The discussion in Fine (2017b) is more pertinent however. In contrast to the Fine (2012b), it deals not just with the purely structural features of ground but also with its interaction with the truth-functional connectives. In contrast to the discussion in Fine (2012a), it explicitly works with a conception of states encompassing merely possible and indeed impossible states rather than just actually obtaining facts, resulting in a much more plausible view.

[32] Cf. (Correia, 2016: p. 107). More accurately, Correia offers the notion of a sentence's *fittingly* describing a situation as a specification of Fine's notion of a sentence being *exactly* verified by a state.

section of the paper, he then proposes a substitutivity principle for the logic of ground that permits the substitution of sentences in an argument place of the grounding operator in case the sentences stand in the relation of factual equivalence, thereby assigning to factual equivalence the role of ground-theoretic equivalence as I have used the term.[33]

Fine (2016) and Correia (2016) determine the logics of ground-theoretic equivalence under a couple of closely related semantic implementations of (TM). Fine obtains a logic that can very naturally be axiomatized as follows:

(Collapse $\neg\neg$)	$A \approx \neg\neg A$
(Collapse $\wedge$)	$A \approx A \wedge A$
(Commutativity $\wedge$)	$A \wedge B \approx B \wedge A$
(Associativity $\wedge$)	$A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$
(Collapse $\vee$)	$A \approx A \vee A$
(Commutativity $\vee$)	$A \vee B \approx B \vee A$
(Associativity $\vee$)	$A \vee (B \vee C) \approx (A \vee B) \vee C$
(De Morgan 1)	$\neg(A \vee B) \approx \neg A \wedge \neg B$
(De Morgan 2)	$\neg(A \wedge B) \approx \neg A \vee \neg B$
(Distributivity 1)	$A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$
(Distributivity 2)	$A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$
(Symmetry)	$A \approx B \ / \ B \approx A$
(Transitivity)	$A \approx B,\ B \approx C \ / \ A \approx C$
(Preservation $\vee$)	$A \approx B \ / \ A \vee C \approx B \vee C$
(Preservation $\wedge$)	$A \approx B \ / \ A \wedge C \approx B \wedge C$

This logic—let us call it $\mathfrak{W}_1$—coincides with R. B. Angell's logic of analytic equivalence (Angell (1977)), which had also been proposed as the logic of ground-theoretic equivalence in Correia (2010). The axiom (Distributivity 2) has since been independently criticized on similar grounds by Krämer and Roski (2015) and (Correia, 2016: p. 119). In the latter paper, Correia therefore drops (Distributivity 2), and proves the resulting weaker logic— call it $\mathfrak{W}_2$—sound and complete with respect to a slightly modified form of the truthmaker semantics used in Fine (2016).

Similar to Fine's looser notions of verification, Correia also discusses looser versions of describing. Again, these are of no importance for our present purposes.

33 Strictly speaking, this is not quite right, since Correia is only committed to the claim that factual equivalence therefore implies ground-theoretic equivalence, not to the converse. To simplify presentation, I shall conduct my discussion under the assumption that the converse claim is accepted, too. However, nothing substantive hangs on this, as my critical comments target exclusively the claim that factual equivalence implies ground-theoretic equivalence.

Although in many ways attractive, the truthmaker account of ground-theoretic equivalence suffers from an important limitation: it is unable to validate the above described introduction and elimination principles for ground—indeed, it is precisely this limitation that provides the motivation for the development in Krämer (2018) of the mode-ified truthmaker account. The most obvious instance of the problem concerns the principle that P grounds $\neg\neg P$: in conjunction with the axiom (Collapse $\neg\neg$), it would enable us to derive $A < A$ for any A, rendering ground not only not irreflexive, but reflexive, which is an absurd result. Similar comments apply to the other two collapse rules (Collapse $\wedge$) and (Collapse $\vee$).

Beyond the three collapse principles, there are four more axioms of $\mathfrak{W}_1$ that are invalid even under the most coarse-grained mode-theoretic account $\mathfrak{E}$, namely the two associativity principles and the two distributivity principles. It is worth considering briefly what may be said in defence of their rejection, since especially the associativity principles may appear quite plausible at first glance. The first and most important point is that we may reject these principles on essentially the same grounds that motivate the rejection of the collapse principles: they are incompatible with the introduction and elimination principles.[34] I shall illustrate the problem for (Associativity $\vee$), the other cases are similar. By the introduction principle for $\vee$, we have $B \vee C < A \vee (B \vee C)$. By (Associativity $\vee$), we could then infer that $B \vee C < (A \vee B) \vee C$. By the elimination principle for $\vee$, it follows that either (a) $B \vee C \leq (A \vee B)$ or (b) $B \vee C \leq C$. But from (b) we could infer, using the introduction principle for $\vee$ and the suitable transitivity principle, that $C < C$, again rendering ground reflexive. From (a), however, we could infer by the same principles that $C < (A \vee B)$, and thus that any disjunction is grounded by everything, which is absurd.

Fine and Correia are of course aware that the proposed account of ground-theoretic equivalence is not compatible with these introduction and elimination principles. Their response is not to simply reject the rules and to insist on the correctness of the truthmaker account, but instead to invoke a distinction between two alternative and equally legitimate conceptions of ground. On a conception of ground as *worldly*, they suggest, the truthmaker account of ground and equivalence is adequate, and the introduction and elimination rules invalid. On a conception of ground as *representational*, in contrast, the

[34] Of course, these principles are not beyond reasonable doubt themselves, and it would be of interest to explore both how they might be modified to allow for Associativity to hold, and whether the mode-theoretic account might be modified accordingly. However, since the introduction and elimination principles are fairly widely accepted and have considerable initial appeal, their incompatibility with Associativity gives us at least some reason to reject Associativity.

opposite is the case: the introduction and elimination rules hold, and the truthmaker account constitutes an insufficiently fine-grained view of ground-theoretic equivalence.[35]

I am happy to grant that there is a legitimate conception of ground, and ground-theoretic equivalence, for which the truthmaker account is adequate. I also maintain, naturally, that there is another legitimate conception of the notions for which the mode-theoretic account is adequate. I shall return to the question of its classification with respect to worldliness or representationality in the final subsection of this section.

4.2. **Correia's Representational Account.** Correia (2017) presents the so far only logic of ground that validates the introduction and elimination principles we stated above, and for which a soundness and completeness results exists. The logic of ground incorporates a logic of what Correia calls *propositional equivalence*, which is the relation that, within his system, plays the role of ground-theoretic equivalence: it obtains just in case the relevant propositions are exactly alike in all respects to which the notion of ground is sensitive.

Transposed to our setting, Correia's account may be described as comprising the following rules, constituting the *representational*[36] system $\mathfrak{R}$:[37]

(Commutativity $\vee$)	$A \vee B \approx B \vee A$
(Commutativity $\wedge$)	$A \wedge B \approx B \wedge A$
(Reflexivity)	$A \approx A$
(Symmetry)	$A \approx B \ / \ B \approx A$
(Transitivity)	$A \approx B,\ B \approx C \ / \ A \approx C$
(Preservation $\vee$)	$A \approx B \ / \ A \vee C \approx B \vee C$
(Preservation $\wedge$)	$A \approx B \ / \ A \wedge C \approx B \wedge C$
(Preservation $\neg$)	$A \approx B \ / \ \neg A \approx \neg B$

[35] The distinction between a worldly and a conceptual or representational conception of ground was first introduced by Correia (2010: p. 256f). In that paper, Correia argued against the representational view, but he has since come to view it as a legitimate alternative conception (cf. e.g. (Correia, 2018: p. 18n16)). For Fine's view of the matter, see esp. (Fine, 2017b: p. 685f).

[36] The label is motivated by the fact that Correia explicitly advocates the logic as adequate relative to a conception of ground as a representational rather than worldly relation.

[37] The background language in Correia (2017) is quite different from $\mathcal{L}_{\approx}$, so the rules he writes down look a bit different. (For instance, his basic expression for ground-theoretic equivalence connects a set of sentences Δ with a single sentence φ, and is supposed to say that Δ is non-empty, and every sentence in Δ is equivalent to φ.) He also adds various elimination rules for ground-theoretic equivalence which are redundant for the purpose of deriving equivalences true in virtue of propositional logical form, though not for the broader purposes pursued in Correia (2017).

This system differs from $\mathfrak{I}$ by dropping the DeMorgan rules and replacing the rule of preservation under double negation by the stronger rule of preservation under single negation. Since $\mathfrak{R}$ lacks some rules of $\mathfrak{I}$ but also contains a rule that is not valid in $\mathfrak{I}$, it is not as straightforward to compare the systems as in the previous cases. Still, it can be shown that $\mathfrak{R}$ yields a strictly more fine-grained conception of ground-theoretic equivalence in the sense that the set of equivalences it proves is a proper subset of the equivalences proved by $\mathfrak{I}$ (the proof is in appendix B).

As indicated above, Correia does not offer an informative general account of ground-theoretic equivalence. It is of course correct to say, on his approach, that sentences S and T are ground-theoretically equivalent iff they express the same proposition, so in this way we can give general necessary and sufficient conditions for ground-theoretic equivalence. But Correia is explicit that the relevant notion of a proposition is to be understood as introduced in a purely functional way: 'they are the items that play the role of the relata of the relation of strict grounding' (Correia, 2017: p. 515). This functional characterization cannot serve to adequately motivate the specific principles Correia lays down with respect to ground-theoretic equivalence, since it is consistent with propositions being mode-theoretic contents, and thus individuated in a more coarse-grained way.[38]

Can Correia's formal account be supplemented by a general account appropriately motivating the former? A natural idea is to offer an account along the following lines. Two sentences are ground-theoretically equivalent iff they express the same proposition. Propositions have internal, quasi-syntactic structure. For example, any conjunction $P \wedge Q$ has a unique decomposition into two immediate constituents: the concept of conjunction and the doubleton set of P and Q. Similarly for disjunction and negation, and perhaps other proposition-forming concepts.

Although such an account could certainly be developed, and appears by no means absurd, I think that there is at least one important way in which it would inevitably be less satisfying than either the mode-theoretic or indeed the truthmaker-theoretic account. On both these approaches, we can see *why* ground should be sensitive to the features that we are taking it to be sensitive to. For the same kinds of semantic features of a sentence are used to say *both* when a statement of ground is true and when sentences are ground-theoretically equivalent. Thus, on the truthmaker-theoretic account, ground is sensitive to what states a given proposition is verified by, because to be (weakly) grounded by another proposition is just to be verified by all the states verifying that proposition. And on the mode-theoretic account, ground is sensitive to the modes in which a proposition

[38] Not that Correia claims otherwise, I hasten to add.

may be verified because for Γ to ground P just is for it to be the case that verifying Γ is a mode of verifying P. Since one cannot in this way give an attractive general account of ground in terms of the quasi-syntactic conceptual structure of propositions, one cannot in this way explain why ground should be sensitive to differences pertaining to that sort of structure.

4.3. **Correia's Relative-Fundamentality-Based Account.** In his recent paper on the logic of relative fundamentality (2018), Correia discusses the possibility of giving a reductive account of ground in terms of an independent notion of relative fundamentality and the notion of (necessary) entailment.[39] The basic idea is that some facts ground another just in case the former necessitate both that the latter obtains and that it is less fundamental than they are. Relative fundamentality is represented via a totally ordered system of *levels*. Each fact is assigned a unique level, and a fact is more fundamental than another just in case it belongs to a lower level. Correia shows that the logic of ground he obtains validates versions of our above introduction rules.[40]

He then points out that within the resulting system, there is a natural way of defining a relation of propositional equivalence, which has the property that whenever two formulas are propositionally equivalent, they are intersubstitutable in the argument places of the grounding operator. In this way, propositional equivalence plays roughly the role of ground-theoretic equivalence.[41] That relation is defined as obtaining between two propositions just in case they are necessarily equivalent and necessarily equally fundamental if true. We may thus consider the following proposal for a general Relative Fundamentality-Based Account of ground-theoretic equivalence (cf. (Correia, 2018: p. 19f)):

RFBA: Sentences S and T are ground-theoretically equivalent iff the propositions expressed by S and T are (i) necessarily equivalent and (ii) necessarily equally fundamental if true.

Although Correia does not determine the exact resulting logic of ground-theoretic equivalence, he does specify for a number of important principles whether they hold or not.

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[39] I should emphasize that Correia does not endorse this account, but only makes the far weaker claim that it merits discussion.

[40] Cf. (Correia, 2018: p. 17f). I say versions of, because they target a factive understanding of ground, and therefore require the truth of the putative grounds as premises.

[41] As in the case of factual equivalence discussed in 4.1, Correia only claims, in effect, that propositional equivalence implies ground-theoretic equivalence, and does not commit to the converse. And as before, for simplicity I focus on the view that results from also endorsing the converse, although again all my critical comments pertain purely to the direction of implication that holds on the account Correia describes.

Indeed, it is not hard to verify that all of the rules of our extensional system turn out to be valid under the interpretation of $\approx$ as expressing the relative-fundamentality-based notion of equivalence.[42]

This equivalence relation is not co-extensional with the mode-theoretically defined one, though: it is strictly wider. Consider, for example, the pair of $\bot \vee (\top \vee \bot')$ and $(\bot \vee \top) \vee \bot'$, where $\top$ is some necessary truth, and $\bot$ and $\bot'$ are distinct necessary falsehoods. Evidently, the two propositions are necessarily true. Moreover, the level of both propositions is necessarily that of $\top$ plus 2. This is because the level of a disjunction with just one true disjunct is defined by Correia to be that of the true disjunct plus 1 (cf. (Correia, 2018: p. 6)). Hence, $\bot \vee (\top \vee \bot')$ and $(\bot \vee \top) \vee \bot'$ are equivalent under Correia's account. But they are inequivalent under the mode-theoretic account, since only the former may be verified via $\top \vee \bot'$, and only the latter via $\bot \vee \top$.

It seems to me that the relative-fundamentality-based account faces serious objections that concern necessary falsehoods and necessary equivalence.[43] Firstly, the account permits no distinction between necessary falsehoods: they are all necessarily equivalent and necessarily equally fundamental if true. As a result, the account also cannot respect some intuitive ground-theoretic distinctions in the realm of truths. For suppose P is true. Then (P or 2+2=5) grounds that (P or 2+2=5) or Q, for arbitrary Q. But it then follows that likewise (P or snow is white and not white) grounds that (P or 2+2=5) or Q, which seems counter-intuitive.

The second problem concerns necessary equivalence, and seems even more serious to me. It is based on the widely held assumptions that the existence of a set is necessary given the existence of its members, and indeed strictly and fully grounded by the existence of its members. It is then also very plausible to take the fact that a certain set exists to belong to level n+1, where n is the highest level occupied by any fact to the effect that a certain member of the set exists. But then consider the following truths:

42 Correia himself mentions the DeMorgan principles and the commutativity principles. The structural rules and preservation rules are unproblematic. It is moreover clear that the collapse principles $A \vee A \approx A \wedge A \approx \neg\neg A$ turn out valid since the relevant propositions are clearly necessarily equivalent and necessarily equally fundamental. It is only slightly harder to verify that the principles I have described as introduction rules for $\leq$, with the abbreviations suitably unpacked, also turn out valid in Correia's system.

43 Of course, this is not to say that the account could not be modified in such a way as to avoid these problems. At the very end of his paper, Correia himself mentions potential problems similar to those discussed here, and hints at a possible refinement of the view that avoids them, and potentially also the ones I describe. The crucial move would be to appeal to a richer conception of worlds that allows both for incomplete and for inconsistent worlds. Still, as far as the version that is given a detailed development in the paper is concerned, my objections apply.

(1) {{Socrates}} exists
(2) {{Socrates}, Socrates} exists

By our assumptions, they are necessarily equivalent and they are necessarily equally fundamental—hence they turn out to be ground-theoretically equivalent. But this is a very implausible result. For instance, only the former truth seems to ground that {{{Socrates}}} exists.[44]

4.4. **Poggiolesi's Account.** The final account I want to consider and compare to my own is that recently developed by Francesca Poggiolesi. In her 2016b, Poggiolesi proposes a definition, in syntactic form and for a standard propositional language, of a relationship of grounding. In 2018, she then specifies a suitable deductive system that she proves to be sound and complete relative to the definition of grounding. As she makes clear, implicit in her account of grounding is a distinctive view of ground-theoretic equivalence. Poggiolesi's account is in some ways very close to my own. In particular, it is characterized by repetition-sensitivity and order-insensitivity, and it still counts as ground-theoretically equivalent certain sentences with different connectives and overall different representational structures. Still, as we shall see, there are also significant differences between our views.

Before addressing specifically the matter of ground-theoretic equivalence, we should note that the notion of *grounding* that Poggiolesi investigates in these papers is somewhat different from the notion I have worked with, and that is in focus in most other recent contributions to the grounding debate. For the notion Poggiolesi discusses is explicitly a notion of *complete* and *immediate formal* grounding (2016b: p. 3150f). Each of the highlighted adjectives requires comment. First, by *formal* grounding, Poggiolesi means what others have called *logical* grounding, i.e. grounding that holds in virtue of the logical forms of the relevant sentences alone. Second, whereas most authors think of grounding as a transitive (or normally transitive) relation, Poggiolesi's focus is on the special case of *immediate* grounding.[45] Third, by a *complete* immediate ground of a truth, Poggiolesi understands the collection of *all* partial immediate grounds of the truth. Thus a complete immediate ground is not the same as a *full* immediate ground in our sense. For instance, if P and Q are both true, then each of them would normally be

44 The same example, given my assumptions, also constitutes a direct objection to the account of ground that Correia describes, for the account then immediately implies that the truth that {{Socrates}, Socrates} exists grounds that {{{Socrates}}} exists. Thanks here to Fabrice Correia for helpful discussion.

45 A more standard notion might then of course be obtained from the immediate one by closing it under a suitable transitivity principle.

seen as a full immediate ground of $P \vee Q$. But only their collection $\{P, Q\}$ is a complete immediate ground of $P \vee Q$.[46] The case of disjunction also illustrates the motivation for another deviation in Poggiolesi's account from previous views of ground, namely that ground is conceived as relative to a background 'robust condition' (2016b: p. 3159ff): P is a complete and immediate formal ground of $P \vee Q$ under the condition that $\neg Q$ holds, but not under the condition that Q holds.[47]

It would take us too far here to explain the details of Poggiolesi's elaborate definition of ground, but we can perhaps convey some of the spirit behind it. The fundamental idea is that logical grounding is a matter of (a) the groundee being derivable from the grounds, (b) the grounds being less complex, in a particular sense, than the groundee, and (c) the negation of the groundee being derivable from the collection of the negation of the grounds, together with the (possibly empty) robust condition. For instance, P grounds $P \vee Q$ under robust condition $\neg Q$ since $\{P, \neg Q\}$ is less complex, in the relevant sense, than $P \vee Q$, $P \vee Q$ is derivable from P, and $\neg(P \vee Q)$ is derivable from $\neg P$ together with the robust condition $\neg Q$.[48]

Despite the differences in the targeted concept of ground, some of the principles Poggiolesi accepts or rejects correspond in a clear way to principles in our own framework, so that a meaningful comparison is possible. In fact, Poggiolesi herself helpfully highlights two significant differences between her view and the views considered in Correia (2014), Fine (2012a), and Schnieder (2011), which in the relevant respects agree with the account defended here (cf. her 2016b: sec. 7, 2018: sections 2, 6). The *first* difference concerns the question of the immediate grounds of a proposition of the form $\neg(\neg P \vee \neg Q)$. On my view, such a proposition is immediately grounded by $\neg\neg P, \neg\neg Q$. On Poggiolesi's view, in contrast, it is not grounded by that collection at all. Instead,

[46] As Poggiolesi highlights, both in imposing an immediacy and a completeness requirement, she is following in the footsteps of Bernard Bolzano, who was the first to develop a general and systematic theory of a notion of grounding that is very close to the contemporary notion(s); see esp. his Bolzano (1837), translated as Bolzano (2014). For a recent book-length study of Bolzano on grounding, see Roski (2017).

[47] Strictly speaking, the condition is not that $\neg Q$ is true, but that what Poggiolesi calls the *converse* of Q is true, which is sometimes but not always identical to $\neg Q$. The difference does not matter for our purposes. See (Poggiolesi, 2016b: p. 3155) for the precise definition of the converse of a formula. (Thanks here to an anonymous referee.)

[48] The caveat of the previous footnote with respect to $\neg Q$ applies here as well. Derivability here is simply classical derivability. The relevant sense of comparative complexity – what Poggiolesi calls 'completely and immediately less g-complex than' – has a somewhat complicated syntactic definition, for the details of which I have to refer the reader to Poggiolesi's article, esp. definition 4.8 on 2016b: p. 3158.

its immediate ground is P, Q. This is connected to a difference explicitly concerning ground-theoretic equivalence. On my view, the De Morgan equivalents $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are ground-theoretically equivalent. On Poggiolesi's view, they are not. The *second* difference concerns the principles of associativity for conjunction and disjunction. On Poggiolesi's view, the propositions $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$ are ground-theoretically equivalent, and likewise for disjunction. On my view, as we have discussed, they are not. Correspondingly, Poggiolesi takes $P, Q \wedge R$ to be an immediate ground of $(P \wedge Q) \wedge R$, whereas I do not.

So like my proposed account, Poggiolesi endorses a more fine-grained picture of ground-theoretic equivalence than the worldly, truthmaker-based account, rejecting, for example, that $A \approx \neg\neg A$ and that $A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$. But whereas my account retains the DeMorgan equivalences and gives up Associativity, Poggiolesi's account gives up DeMorgan and retains Associativity. This lends to her view a very distinctive and novel character, and it would be interesting to see if something like the present mode-theoretic framework could also be used to provide a semantics for her view.

I know of no decisive reasons to prefer my account to Poggiolesi's, or hers over mine. The combination of principles accepted under my approach, in conjunction with the introduction and elimination principles for ground, strikes me as considerably more intuitive than Poggiolesi's. But Poggiolesi has different intuitions (cf. her 2016b: p. 3165f), and so appeals to intuitions do not seem to provide a way forward here. A justified choice between the accounts, I suspect, will have to be based on a comprehensive evaluation of their overall theoretical virtues, which it is beyond the scope of this paper to carry out.[49]

However, Poggiolesi's work also contains the material for a direct challenge to my account. For I have not only rejected the associativity principles, I have also endorsed the principles of *commutativity* as principles of ground-theoretic equivalence. And the discussion in Poggiolesi (2016b: p. 3156) strongly suggests that she takes these two kinds of principles to stand and fall together. It is therefore worth considering whether

[49] Here are two considerations that would seem relevant to such an evaluation. Poggiolesi (2016a) points out that her account underwrites certain connections between ground and the normality of proofs. To the extent that it is indepedently plausible that there should be such connections, this would be a point in favour of her account. On the other hand, Poggiolesi's logic of ground appears not to be closed under substitution, which is also a property that is often seen as desirable feature of a logic. (Although $\neg(\neg P \vee \neg Q)$ is not grounded by $\neg\neg P, \neg\neg Q$, when P and Q are not themselves negations, then even on Poggiolesi's account, $\neg(P \vee Q)$ *is* grounded by $\neg P$, $\neg Q$.) So if a case can be made that the logic of ground should be closed under substitution, this would favour the present account.

specifically the combination of the commutativity principles with the rejection of the associativity principles is problematic.

Why might one take commutativity and associativity to stand and fall together? One idea might be that both simply amount to an insensitivity to order in the relevant expressions.[50] As I have stressed, the sentences $A \wedge B$ and $B \wedge A$ differ only in the order of the arguments of $\wedge$. Then perhaps in the case of $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$, one might say the difference is purely whether $\wedge$ is first applied to B and C, and then also to A, or first to A and B, and then also to C. But this is a misleading description of the case. For it is *not* the case that in both sentences, the same formulas show up as arguments of $\wedge$, only in a different order. In $A \wedge (B \wedge C)$, the complex formula $B \wedge C$ is an argument of $\wedge$, whereas in $(A \wedge B) \wedge C$, that formula does not occur at all. And it is just this difference that is reflected, on my view, in the grounds of the propositions expressed by the formulas: the proposition expressed by $B \wedge C$ is a partial ground of the proposition expressed by $A \wedge (B \wedge C)$, but not of the proposition expressed by $(A \wedge B) \wedge C$. Of course, one could try to argue that this difference should not be reflected ground-theoretically. But one cannot do this simply on the grounds that the differences between $A \wedge B$ and $B \wedge A$ should not be reflected ground-theoretically. These are a different *kind* of difference.

Another idea might be that with respect to both commutativity and associativity, the crucial point is that the differences between the relevant pairs of sentences are purely *notational*, and thus not the kind of difference to which ground can plausibly be considered sensitive.[51] Thus, in the case of $A \wedge B$ and $B \wedge A$, one might say that these are merely alternative notations for the application of conjunction to *one and the same unordered pair* of contents. In the case of $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$, the claim might then be that these are merely alternative notations for the application of conjunction to *one and the same unordered triple* of contents. However, on closer inspection, the second claim is not plausible. For $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ both involve *two* applications of conjunction, and each of these applications is to a pair, not a triple. This is simply a consequence of the grammar of our standard propositional language, in which $\wedge$ is a two-place operator.

Still, one may feel that there is something to the thought that somehow $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ are just alternative notations for the same thing. Fortunately, this can be explained in a different way, compatible with the rejection of the associativity principles. For suppose that one thinks that each of A, B, and C are true, and simply wishes to say as much. It then feels artificial to have to choose between $(A \wedge B) \wedge C$ and $A \wedge$

[50] This was suggested to me by an anonymous referee.

[51] Thanks to two anonymous referees for pressing me on this point.

$(B \wedge C)$ as vehicles for saying what one wants to say. Relative to one's communicative intentions, the difference between the two is irrelevant. However, it doesn't follow that there are not ground-theoretic differences between the contents of $(A \wedge B) \wedge C$ and $A \wedge (B \wedge C)$. It may simply be that relative to one's communicative goals, these ground-theoretic differences, too, are irrelevant. Moreover, if it is seen as a bad thing that one has to choose between different, ground-theoretically inequivalent sentences to say what one wants to say, what this seems to suggest is that one should use a different *language*, namely one with a conjunction-operator $\bigwedge$ operating on a set or multi-set. Then one could simply utter $\bigwedge\{A, B, C\}$, without needing to make an arbitrary choice between alternative sentences. I think there are advantages to using such a language, but to facilitate comparison with previous approaches, I here focus on a more standard language with only binary conjunction and disjunction operators.

Finally, some may simply find it highly counter-intuitive that conjunction and disjunction should be commutative without being associative with respect to ground-theoretic equivalence.[52] So to close this discussion, let me try to offer an analogy that may help make the rejection of associativity more intelligible to such an opponent. I think there is a very natural rough picture of how truth-functional operations interact with the ground-theoretic hierarchy which should lead one to *expect* that associativity, but not commutativity, will fail.[53] Consider the set-theoretic operation of *pairing*, mapping arguments x and y to the set $\{x, y\}$. Note that it is commutative but not associative: if $x \neq z$, then $\{x, \{y, z\}\} \neq \{\{x, y\}, z\}$. Now one natural picture has it that the truth-functions behave with respect to the ground-theoretic hierarchy somewhat analogously to how pairing behaves with respect to the set-theoretic hierarchy. In particular, there is an analogous reason for the failure of associativity: it lies in the fact that, roughly speaking, each application of the relevant function involves "raising the level"—either set-theoretic or ground-theoretic—of the arguments of the function. If a function f has this level-raising character, it is easy to see how associativity may fail, for in $f(f(x, y), z)$ the arguments x and y are raised twice, as it were, and z is raised only once, whereas in $f(x, f(y, z))$,

52 An anonymous reviewer has expressed this sentiment.

53 I do not mean to deny that there are also natural rough pictures which would not lead one to expect this. Indeed, I believe there are such alternative pictures. But that is okay, for my aim here is not to establish that commutativity holds and associativity not. It is merely to show that one may *reasonably* take this to be so, and thus that a view of this sort should be considered a serious contender.

x is raised only once, and y and z twice.[54] Now if one thinks of ground and the truth-functions on something like this level-raising model, then it does not seem at all counter-intuitive to reject associativity and accept commutativity. For the level-raising nature of the truth-functions naturally leads to a violation of associativity, without presenting any threat whatsoever to commutativity.

4.5. **Worldly vs. Representational Ground.** Finally, I want to briefly examine whether the mode-theoretic account justifies a view of the conception of ground it is adequate for as *representational*. The answer to this naturally depends on how the worldly/representational distinction is spelled out. I think there are at least three prima facie natural ways of doing this. On the first, ground is worldly (representational) just in case the *relata* of ground are worldly (representational) entities.[55] If we consider the mode-theoretic account as a guide to the nature of the relata of ground, then it would appear justified to say that it yields a representational conception of ground. For the relata of ground are then mode-theoretic propositions, which are clearly representational entities—roughly speaking, they represent that at least one of their modes is actual. But then by parallel reasoning it seems that if we consider the truthmaker-theoretic account as a guide, we would also appear to be justified in saying that it yields a representational conception of ground. For here, too, the relata are clearly representational entities, representing that at least one of their verifiers obtains.

The second explication tracks not the worldliness or otherwise of the relata of ground, but the worldliness or otherwise of the differences between the relata that ground is sensitive to.[56] Since a difference in verifying states is plausibly seen as a worldly difference between propositions—it concerns purely how the proposition relates to worldly entities—, the truthmaker account is then classified as worldly. A difference in modes of verification, in contrast, is not plausibly seen as worldly, since it typically concerns how the proposition relates to other propositions as well as the world (the exception being a difference in direct modes of verification). The mode-theoretic account would accordingly be counted as representational.

The third explication appeals to whether the relata of ground are individuated in terms of representational structure or not. Under that explication, even the mode-theoretic

54 I am not claiming that it is *impossible* to define a level-raising associative function. But starting from the idea of a level-raising combination like pairing, one very *naturally* ends up with a non-associative function.

55 This explication fits nicely with a number of formulations found in Correia's papers, cf. e.g. (Correia, 2010: p. 256f), (Correia, 2017: p. 508).

56 This explication was suggested to me by [blinded].

views plausibly qualify as worldly. For even under the repetition-sensitive conception of modes, since the DeMorgan rules hold for ground-theoretic equivalence, sentences with significantly different representational structures are still seen as ground-theoretically equivalent. In particular, as the case illustrates, the account does not permit a exclusive division between negative and positive propositions, as one would expect of an account that individuated propositions in terms of representational structure. Under the repetition-insensitive conception, not even an exclusive division between conjunctive and disjunctive propositions can be made, since no distinction is made between self-conjunctions and self-disjunctions.

How should we choose between these explications? Clearly, an explication will be unsatisfactory if it yields a distinction that carries no theoretical significance. If the worldly/representational distinction is to have any theoretical interest, it must presumably be correlated with a difference in theoretical roles that the conceptions of ground either side of the divide are suited to play. I have no firm view of the matter, and I lack the space to pursue it here. It may well be, however, that all of the dividing lines just sketched correspond to a difference in plausible theoretical roles. In that case, the most informative description of the status of the mode-theoretic accounts is that they occupy an intermediate position in regard to worldliness between the truthmaker account and the representational account of Correia's.

Appendix A. Soundness and Completeness

A.1. **Definitions.** Recall the definition of $\mathcal{L}_{\approx}$ as based on a propositional language with connectives $\wedge$, $\vee$, and $\neg$, augmented by all expressions of the form $A \approx B$ where $A,\ B$ are sentences of the propositional language (and thus do not already include occurrences of $\approx$). We call expressions of the latter form *equivalences*, and reserve the label 'sentence' for the purely truth-functional sentences. The language comprising only the sentences of $\mathcal{L}_{\approx}$ will be called $\mathcal{L}_B$.

For ease of reference, we repeat some relevant definitions from Krämer (2018):

Definition 1. (Mode-Space) *A* mode-space *is a pair $\langle M, V\rangle$ such that*

(1) M is a non-empty set
(2) V is a non-empty, partial function taking non-empty countable sequences of non-empty subsets of M into members of M
(3) the domain of V is closed under non-empty subsequences and countable concatenation of sequences
(4) $V(\gamma_1^\frown\gamma_2^\frown\ldots) = V(\delta_1^\frown\delta_2^\frown\ldots)$ whenever $V(\gamma_1) = V(\delta_1), V(\gamma_2) = V(\delta_2),\ldots$

Informally, M is the set of modes. Any non-empty set of modes is a proposition. V is the via-function, mapping some sequences of propositions $P_1, P_2, \ldots$ to the mode of verifying via $P_1, P_2, \ldots$. Modes which are never the value of V are called *fundamental*, and their set is denoted M^F. All other modes are called derivative and their set is denoted M^D. Of any indirect modes m and n we can form the fusion $m \sqcup n$ which is $V\langle P_1, P_2, \ldots, Q_1, Q_2, \ldots\rangle$ when $V\langle P_1, P_2, \ldots\rangle = m$ and $V\langle Q_1, Q_2, \ldots\rangle = n$. If V is defined for $\langle P\rangle$, P is called *raisable*, and the set of raisable contents is denoted $\mathcal{R}$.

Conjunction, disjunction, and negation are now defined as follows. First, we define binary operations of fusion ($\sqcup$) and (disjunctive) addition (+) for propositions that are subsets of M^D:

$$P \sqcup Q \quad := \{m \sqcup n : m \in P \text{ and } n \in Q\}$$
$$P + Q \quad := (P \cup Q) \cup (P \sqcup Q)$$

Next, for raisable contents P, we define an operation of *raising* on unilateral contents, which given an input P yields a content $\uparrow P$ just like P except in that it may be verified via P, and in terms of raising, fusion, and addition, we define conjunction and disjunction on raisable unilateral contents:

$$\uparrow P \quad := \{V\langle P\rangle\} \text{ if } P \cap M^D \text{ is empty, } \{V\langle P\rangle\} + (P \cap M^D) \text{ otherwise}$$
$$P \wedge Q \quad := \uparrow P \sqcup \uparrow Q$$
$$P \vee Q \quad := \uparrow P + \uparrow Q$$

Finally, we define conjunction, disjunction, and negation on bilateral contents in $\mathcal{R} \times \mathcal{R}$:

$$\neg \mathbf{P} \quad := \langle \mathbf{P}^-, \uparrow \mathbf{P}^+\rangle$$
$$\mathbf{P} \wedge \mathbf{Q} \quad := \langle \mathbf{P}^+ \wedge \mathbf{Q}^+, \mathbf{P}^- \vee \mathbf{Q}^-\rangle$$
$$\mathbf{P} \vee \mathbf{Q} \quad := \langle \mathbf{P}^+ \vee \mathbf{Q}^+, \mathbf{P}^- \wedge \mathbf{Q}^-\rangle$$

For the operations so defined to behave as desired, the background mode-space needs to satisfy two important conditions:

Definition 2. *A mode-space $\langle M, V\rangle$ is called* complete *iff $P \sqcup Q \in \mathcal{R}$, $P + Q \in \mathcal{R}$, and $\uparrow P \in \mathcal{R}$ whenever $P, Q \in \mathcal{R}$.*

Definition 3. *A mode-space $\langle M, V\rangle$ is called* constrained *iff $V(\gamma) = V(\delta)$ only if the same ground-set corresponds to γ and δ.*

Note that in a constrained mode-space, every derivative mode m corresponds to a unique ground-set, which is denoted by $|m|$. We shall henceforth deal only with complete and constrained mode-spaces.

Definition 4. *A unilateral proposition P is*

- irreflexive *iff: P does not occur in γ whenever $V(\gamma) \in P$,*

- closed *iff: P contains a mode with ground-set* $\Gamma_1 \cup \Gamma_2 \cup \ldots$ *whenever P contains modes with ground-sets* $\Gamma_1, \Gamma_2, \ldots$,
- transitive *iff: P includes a mode with ground-set* Γ, Δ *whenever P includes a mode with ground-set* Δ, *Q and Q includes a mode with ground-set* Γ.
- normal *iff: irreflexive, closed, and transitive.*

We define the two classes of mode-spaces with respect to which we shall establish soundness and completeness results.

Definition 5. *A mode-space* $\langle M, V\rangle$ *is*

- intermediate *iff:* $V(\gamma) = V(\delta)$ *whenever the same multi-set underlies* γ *and* δ
- extensional *iff:* $V(\gamma) = V(\delta)$ *whenever the same set underlies* γ *and* δ

Note that the class of intermediate mode-spaces is exactly the class of mode-spaces compatible with an order-insensitive but repetition-sensitive conception of modes, whereas the class of extensional mode-spaces correspondingly reflects the repetition-insensitive conception of modes.

We are now in a position to give a mode-space *semantics* for $\mathcal{L}_\approx$ by defining the notions of a model, truth in a model, and validity in a class of models.

Definition 6. *If* $\langle M, V\rangle$ *is a mode-space, then* $\mathcal{M} = \langle M, V, [\cdot]\rangle$ *is a* model based on $\langle M, V\rangle$ *just in case* $[\cdot]$ *is a function mapping every sentence in* $\mathcal{L}_\approx$ *to a non-empty subset of M so that for all sentences* $A, B \in \mathcal{L}_\approx$*:*

- $[\neg A] = \neg[A]$
- $[A \wedge B] = [A] \wedge [B]$
- $[A \vee B] = [A] \vee [B]$

$\mathcal{M}$ *is a model just in case* $\mathcal{M}$ *is a model based on some mode-space.*

Definition 7. *(Truth and Validity) For any equivalence* $A \approx B \in \mathcal{L}_\approx$*:*

- $A \approx B$ *is* true in a model $\mathcal{M}$ *(*$\mathcal{M} \models A \approx B$*) iff* $[A] \approx [B]$
- $A \approx B$ *is* valid in a class of models C *(*$\models_C A \approx B$*) iff true in every model in* C.

A.2. **Preparatory Results.** For ease of reference, we repeat the central theorems of Krämer (2018) that we shall need. Throughout, we tacitly restrict attention to complete and constrained mode-spaces.

Lemma 1. *The introduction and elimination principles for bilateral propositions hold (theorems 5 and 6):*

(<∧): $\mathbf{\Gamma} < \mathbf{P} \wedge \mathbf{Q}$ *iff* $\mathbf{\Gamma} \leq \{\mathbf{P}, \mathbf{Q}\}$
(<∨): $\mathbf{\Gamma} < \mathbf{P} \vee \mathbf{Q}$ *iff* $\mathbf{\Gamma} \leq \mathbf{P}$ *or* $\mathbf{\Gamma} \leq \mathbf{Q}$ *or* $\mathbf{\Gamma} \leq \{\mathbf{P}, \mathbf{Q}\}$

($<\neg\neg$): $\mathbf{\Gamma} < \neg\neg\mathbf{P}$ *iff* $\mathbf{\Gamma} \leq \mathbf{P}$
($<\neg\wedge$): $\mathbf{\Gamma} < \neg(\mathbf{P} \wedge \mathbf{Q})$ *iff* $\mathbf{\Gamma} \leq \neg\mathbf{P}$ *or* $\mathbf{\Gamma} \leq \neg\mathbf{Q}$ *or* $\mathbf{\Gamma} \leq \{\neg\mathbf{P}, \neg\mathbf{Q}\}$
($<\neg\vee$): $\mathbf{\Gamma} < \neg(\mathbf{P} \vee \mathbf{Q})$ *iff* $\mathbf{\Gamma} \leq \{\neg\mathbf{P}, \neg\mathbf{Q}\}$

Lemma 2. *The following structural principles hold for normal unilateral propositions (theorem 7):*

(1) $P \not< P$
(2) *If* $\Gamma_1 < P, \Gamma_2 < P, \ldots,$ *then* $\Gamma_1, \Gamma_2, \ldots < P$.
(3) *If* $\Gamma \leq P$ *and* $Q \prec P$ *for all* $Q \in \Gamma$, *then* $\Gamma < P$.
(4) *If* $\Gamma, P \leq P$, *then* $\Gamma \leq P$.
(5) *If* $\Gamma_1 \leq P, \Gamma_2 \leq P, \ldots,$ *then* $\Gamma_1 \cup \Gamma_2 \cup \ldots \leq P$.
(6) *If* $\Gamma < P$ *and* $\Delta, P < Q$, *then* $\Gamma, \Delta < Q$.
(7) *If* $\Gamma_1 \leq P_1, \Gamma_2 \leq P_2, \ldots,$ *and* $P_1, P_2, \ldots \leq Q$ *then* $\Gamma_1, \Gamma_2, \ldots \leq Q$
(8) *If* $P \leq Q$ *and* $Q \prec R$ *then* $P \prec R$
(9) *If* $P \prec Q$ *and* $Q \leq R$ *then* $P \prec R$
(10) *If* $P \leq Q$ *and* $Q \leq R$ *then* $P \leq R$

Lemma 3. *Normality of unilateral propositions is preserved under* $\wedge$, $\vee$, *and* $\uparrow$ *(theorem 8).*

We state without proof the following straightforward facts about ground-theoretic equivalence:

Lemma 4. $\approx$ *has the following properties:*

(1) $\approx$ *is an equivalence relation—for all* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ *we have (i)* $\mathbf{P} \approx \mathbf{P}$, *(ii) if* $\mathbf{P} \approx \mathbf{Q}$ *then* $\mathbf{Q} \approx \mathbf{P}$, *and (iii) if* $\mathbf{P} \approx \mathbf{Q}$ *and* $\mathbf{Q} \approx \mathbf{R}$ *then* $\mathbf{Q} \approx \mathbf{R}$
(2) $\approx$ *is preserved under conjunction, disjunction, and double negation—for all* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$, *if* $\mathbf{P} \approx \mathbf{Q}$, *then (i)* $\mathbf{P} \wedge \mathbf{R} \approx \mathbf{Q} \wedge \mathbf{R}$, *(ii)* $\mathbf{P} \vee \mathbf{R} \approx \mathbf{Q} \vee \mathbf{R}$, *and (iii)* $\neg\neg\mathbf{P} \approx \neg\neg\mathbf{Q}$
(3) $\approx$ *satisfies the DeMorgan rules—for all* $\mathbf{P}, \mathbf{Q}$: *(i)* $\neg(\mathbf{P} \wedge \mathbf{Q}) \approx \neg\mathbf{P} \vee \neg\mathbf{Q}$ *and (ii)* $\neg(\mathbf{P} \vee \mathbf{Q}) \approx \neg\mathbf{P} \wedge \neg\mathbf{Q}$

We now establish some easy *sufficient* identity conditions for unilateral contents:

Lemma 5. *Let* P, Q *be raisable unilateral propositions.*

(i) *If the mode-space is intermediate:* $P \wedge Q = Q \wedge P$ *and* $P \vee Q = Q \vee P$.
(ii) *If the mode-space is extensional:* $P \wedge P = P \vee P$.
(iii) *If the mode-space is extensional and* P *closed:* $P \vee P = \uparrow P$

Proof. For (i): We show that the operation of fusion on the modes is commutative. Let m and n be derivative modes, and suppose $m = V(\gamma)$ and $n = V(\delta)$. By the definition of fusion, $m \sqcup n = V(\gamma^\frown\delta)$ and $n \sqcup m = V(\delta^\frown\gamma)$. Since the mode-space is assumed to be intermediate, V maps two sequences to the same mode if they correspond to the same multi-set, and since $\gamma^\frown\delta$ and $\delta^\frown\gamma$ correspond to the same multi-set, it follows that $m \sqcup n = n \sqcup m$. From this, the result follows immediately by definition of $\wedge$ and $\vee$.

For (ii): By application of the definitons, $P \wedge P = \uparrow P \sqcup \uparrow P \subseteq \uparrow P \cup \uparrow P \cup (\uparrow P \sqcup \uparrow P) = \uparrow P + \uparrow P = P \vee P$. It remains to show that $\uparrow P \subseteq \uparrow P \sqcup \uparrow P$. This follows from the idempotence of $\sqcup$ as defined on derivative modes in extensional mode-spaces. Let m be a derivative mode and assume $m = V(\gamma)$. Then $m \sqcup m = V(\gamma^\frown\gamma)$ and since the same set underlies γ and $\gamma^\frown\gamma$, by extensionality of the mode-space, $m = V(\gamma) = V(\gamma^\frown\gamma) = m \sqcup m$.

For (iii): $\uparrow P \subseteq P \vee P$ is immediate by definition of $\vee$. It remains to show that $\uparrow P \sqcup \uparrow P \subseteq \uparrow P$. So let $m \in \uparrow P \sqcup \uparrow P$. Then for some m_1, m_2: $m = m_1 \sqcup m_2$ and $m_1 \in \uparrow P$ and $m_2 \in \uparrow P$. By assumption, P is closed. It is straightforward to show that $\uparrow P$ is then also closed, and so $\uparrow P$ contains some mode with ground-set $|m_1| \cup |m_2| = |m_1 \sqcup m_2| = |m|$. Since in an extensional mode-space, no two modes have the same ground-set, it follows that $m \in \uparrow P$ and thus $P \vee P \subseteq \uparrow P$. □

We may also establish some substantive *necessary* conditions for certain kinds of unilateral propositions to be identical:

Lemma 6. *Let P, Q, R, S be normal raisable unilateral propositions.*

(i) $\uparrow P = \uparrow Q$ *implies* $P = Q$
(ii) $P \wedge Q = \uparrow R$ *implies* $P = Q = R$
(iii) $P \vee Q = \uparrow R$ *implies* ($P = R$ *and* $Q \leq R$) *or* ($Q = R$ *and* $P \leq R$)
(iv) $P \wedge Q = R \wedge S$ *implies* $\{P, Q\} = \{R, S\}$
(v) $P \wedge Q = R \vee S$ *implies* $P = Q$ *and* (($R = P$ *and* $S \leq R$) *or* ($S = P$ *and* $R \leq S$))
(vi) $P \vee Q = R \vee S$ *implies that either*
  a. $\{P, Q\} = \{R, S\}$, *or*
  b. $P = R$ *and* $Q \leq P$ *and* $S \leq R$, *or*
  c. $P = S$ *and* $Q \leq P$ *and* $R \leq S$, *or*
  d. $Q = R$ *and* $P \leq Q$ *and* $S \leq R$, *or*
  e. $Q = S$ *and* $P \leq Q$ *and* $R \leq S$

Proof. By use of the equivalences noted at the beginning of this section and the transitivity and antisymmetry of $\leq$ and $\preceq$.

For (i): Since $P < \uparrow P$, it follows from the antecedent that $P < \uparrow Q$ and hence $P \leq Q$. Likewise since $Q < \uparrow Q$, it follows that $Q < \uparrow P$ and hence $Q \leq P$. By antisymmetry of $\leq$, $P = Q$.

For (ii): Since $P, Q < P \wedge Q$, it follows from the antecedent that $P, Q <\uparrow R$ and hence $P, Q \leq R$. So $P \leq R$ and $Q \leq R$. Moreover, $R <\uparrow R$ so $R < P \wedge Q$, so $R \leq \{P, Q\}$. It follows that $R \leq P$ and $R \leq Q$, so $R \leq P$ and $R \leq Q$, and hence by antisymmetry of $\leq$ that $R = P$ and $R = Q$.

For (iii): In similar fashion as before, it follows from the antecedent that $P \leq R$ and $Q \leq R$, as well as that either (a) $R \leq P$ or (b) $R \leq Q$. If (a), then by antisymmetry of $\leq$ we have $P = R$, and if (b), we have $Q = R$.

For (iv): From the antecedent it is straightforward to show (1) that either (1a) $P \leq R$ and $Q \leq S$ or (1b) $P \leq S$ and $Q \leq R$ and (2) that either (2a) $R \leq P$ and $S \leq Q$ or (2b) $R \leq Q$ and $S \leq P$. If (1a) and (2a), then by antisymmetry of $\leq$, $P = R$ and $Q = S$ follows. If (1b) and (2b), then it follows that $P = S$ and $Q = R$. If (1b) and (2a), we have $P \leq S \leq Q \leq R \leq P$ and so $P = Q = R = S$. Likewise if (1a) and (2b), we have $P \leq R \leq Q \leq S \leq P$ and so again $P = Q = R = S$. So in all four cases, $\{P, Q\} = \{R, S\}$.

For (v): From the antecedent it follows that $R \leq P$ and $R \leq Q$ and $S \leq P$ and $S \leq Q$. Moreover, either (a) $P, Q \leq R$ or (b) $P, Q \leq S$ or (c) $P, Q \leq \{R, S\}$. If (a), then $P \leq R$ and so $P = R$. Similarly $Q \leq R$, and so $Q = R$. Hence $S \leq P = Q = R$, establishing the consequent. If (b), then $P \leq S$, so $P = S$, and $Q \leq S$, so $Q = S$. Hence $R \leq P = Q = S$, again establishing the consequent. Finally, if (c), then it easy to show that either $P \leq R$ and $Q \leq S$ or $P \leq S$ and $Q \leq R$. In the first case, $P = R \leq Q = S \leq P$, so $P = Q = R = S$. In the second case, $P = S \leq Q = R \leq P$, so again $P = Q = R = S$. Either way, the consequent is again established.

For (vi): From the antecedent it follows (1) that (1a) $P \leq R$ or (1b) $P \leq S$, and (2) that (2a) $Q \leq R$ or (2b) $Q \leq S$, and (3) that (3a) $R \leq P$ or (3b) $R \leq Q$, and (4) that (4a) $S \leq P$ or (4b) $S \leq Q$. It can now be shown that under each of the 16 possible combinations, one of the conditions a.–e. obtain. We shall restrict ourselves to an illustrative four cases; the remaining ones follow the same pattern. Suppose first that (1a), (2a), (3a), and (4a) obtain. Then by (1a) and (3a), $P = R$. By (2a), $Q \leq R = P$. By (4a), $S \leq P = R$. So case b. above obtains. Suppose now that instead of (4a), (4b) obtains. We still have $P = R$ and $Q \leq P$ as before. By (4b), $S \leq Q$, and since $Q \leq P$, we obtain $S \leq P = R$, as required for case b. For a different sort of case, suppose (1a), (2b), (3a), and (4b) obtain. Then still $P = R$, and by (2b), $Q \leq S$, as well as by (4b) $S \leq Q$, so $S = Q$. It follows that case a. above obtains. Finally, suppose that instead of (3a) and (4b), we have (3b) and (4a). Then $P \leq R$, $Q \leq S$, $R \leq Q$, and $S \leq P$. That is, $P \leq R \leq Q \leq S \leq P$, and thus $P = R = Q = S$. Again it follows that case a. obtains. □

Finally, in an extensional mode-space, binary weak full ground (among unilateral contents) can be characterized in terms of disjunction and identity:

Lemma 7. *For P, Q normal raisable unilateral propositions in an extensional mode-space: $P \leq Q$ iff $P \vee Q = Q \vee Q$.*

Proof. For the right-to-left-direction, assume $P \vee Q = Q \vee Q$. Then $P < P \vee Q$ so $P < Q \vee Q$, so $P \leq Q$. For the left-to-right direction, assume $P \leq Q$. Then either $P = Q$ or $P < Q$. If $P = Q$, then evidently $P \vee Q = Q \vee Q$. So suppose $P < Q$. Given extensionality, to show that $P \vee Q = Q \vee Q$ it suffices to show for arbitrary Γ that $\Gamma < P \vee Q$ iff $\Gamma < Q \vee Q$. But if $\Gamma < Q \vee Q$, then $\Gamma \leq Q$, so $\Gamma < P \vee Q$. If $\Gamma < P \vee Q$, then either (a) $\Gamma \leq P$, or (b) $\Gamma \leq Q$, or (c) $\Gamma \leq \{P, Q\}$, so $\Gamma_P \leq P$ and $\Gamma_Q \leq Q$ for some Γ_P, Γ_Q with $\Gamma = \Gamma_P \cup \Gamma_Q$. If (a), then since $P < Q$ and $Q < Q \vee Q$, it is easy to show that $\Gamma < Q \vee Q$. if (b), then since $Q < Q \vee Q$, it is equally straightforward that $\Gamma < Q \vee Q$. If (c), then by the previous reasoning, $\Gamma_P \leq Q$ and $\Gamma_Q \leq Q$ and so $\Gamma_P \cup \Gamma_Q = \Gamma < Q \vee Q$. □

A.3. **Adequacy of the Intermediate System.** For φ an equivalence in $\mathcal{L}_\approx$, we write $\vdash_\Im \varphi$ to say that φ is derivable within $\Im$, and we write $\models_\Im \varphi$ to say that φ is valid in the class of models based on complete, constrained, intermediate mode-spaces. We show first that $\Im$ is sound with respect to that class of models.

Theorem 8. (Soundness of the Intermediate System) $\models_\Im \varphi$ *whenever* $\vdash_\Im \varphi$.

Proof. The soundness of (Comm.$\vee$) and (Comm.$\wedge$) is immediate from lemma 5(i). The soundness of (Reflexivity), (Symmetry), (Transitivity), the preservation rules and the DeMorgan rules is immediate from lemma 4. □

To prove completeness, we construct a canonical intermediate mode-space and model, and show that every equivalence that is true in this model is derivable within $\Im$. The canonical mode-space is defined as follows.

Definition 8. *The* canonical intermediate mode-space *for $\mathcal{L}_\approx$ is the pair $\langle M_I, V_I \rangle$, where*

- $M_0 := \{A \in \mathcal{L}_\approx : A \text{ is a literal}\}$
- $C_n := \wp(M_n) \backslash \{\emptyset\}$
- $M_{n+1} := \{m : m \in M_0$ *or m is a non-empty multi-set of members of* $C_n\}$
- $C := \bigcup_{n \in N} C_n$
- $M_I := \{m : m \in M_0$ *or m is a non-empty multi-set of members of* $C\}$
- $V_I(\gamma) = \Gamma$ *if γ is a non-empty countable sequence of members of C and Γ is the underlying multi-set.*
- $V_I(\gamma)$ *is undefined otherwise.*

We now establish that $\langle M_I, V_I \rangle$ is indeed a complete, constrained, intermediate mode-space. We show first that it is a mode-space.

Lemma 9. *$\langle M_I, V_I \rangle$ is a mode-space.*

Proof. First, M_I is non-empty, since M_I includes all the literals in $\mathcal{L}_{\approx}$.

Second, V_I is a non-empty partial function that maps non-empty sequences of subsets of M_I to members of M_I. For since there are non-empty countable sequences of members of C and corresponding underlying multi-sets, V_I is non-empty. Since the members of C are subsets of M_I, the arguments of V_I are sequences of subsets of M_I. Since any multi-set underlying a non-empty countable sequence of members of C is a non-empty multi-set of members of C, the values of V_I are subsets are members of M_I.

Third, the domain of V_I is closed under non-empty subsequences and countable concatenations, for non-empty subsequences and countable concatenations of non-empty countable sequences of members of C are themselves such sequences.

Finally, $V_I(\gamma_1^\frown\gamma_2^\frown\ldots) = V_I(\delta_1^\frown\delta_2^\frown\ldots)$ whenever $V_I(\gamma_1) = V_I(\delta_1)$, $V_I(\gamma_2) = V_I(\delta_2), \ldots$ For the multi-set underlying $\gamma_1^\frown\gamma_2^\frown\ldots$ is determined by which items occur how many times in γ_1, and in γ_2, and ..., which is to say that it is determined by the multi-sets underlying $\gamma_1, \gamma_2, \ldots$. Since the multi-sets corresponding underlying $\gamma_1, \gamma_2, \ldots$ are the same as those underlying $\delta_1, \delta_2, \ldots$ whenever $V_I(\gamma_1) = V_I(\delta_1), V_I(\gamma_2) = V_I(\delta_2), \ldots$, the result is then immediate. □

Next, we show that the recursive construction of the canonical mode-space is cumulative in the following sense:

Lemma 10. *In the construction of $\langle M_I, V_I \rangle$, for all n, $M_n \subseteq M_{n+1}$ and $C_n \subseteq C_{n+1}$.*

Proof. Suppose $m \in M_0$. Then by definition, $m \in M_1$. Suppose $P \in C_0$. Then $\emptyset \subset P \subseteq M_0 \subseteq M_1$ and hence $P \in C_1$. Now assume that the claim holds up to n. Suppose $m \in M_{n+1}$. Then either $m \in M_0$, in which case $m \in M_{n+2}$ follows by definition, or m is a non-empty multi-set of members of C_n. By IH, $C_n \subseteq C_{n+1}$, hence m is a non-empty multi-set of members of C_{n+1}. By definition of M_{n+2}, it follows that $m \in M_{n+2}$. Suppose finally that $P \in C_{n+1}$. Then just as before, $\emptyset \subset P \subseteq M_{n+1} \subseteq M_{n+2}$, hence $P \in C_{n+2}$. □

Lemma 11. *The mode-space $\langle M_I, V_I \rangle$ is complete, constrained, and intermediate.*

Proof. Complete: We show first that $P \sqcup Q$ and $P + Q$ are raisable whenever P and Q are derivative, raisable contents. Since P and Q are derivative contents, $P \in C_n$ for some $n > 0$ and $Q \in C_m$ for some $m > 0$. Now let $j = max(m, n)$. By lemma 10, P and Q are both in C_j and therefore non-empty subsets of M_j. But then it follows that $P \sqcup Q \in \mathcal{R}$. For suppose $m_1 \in P$ and $m_2 \in Q$, so $m_1, m_2 \in M_j$. Then since every M_n with $n > 0$ is closed under fusion, $m_1 \sqcup m_2 \in M_j$. It follows that $P \sqcup Q \subseteq M_j$, and thus that $\{P \sqcup Q\} \subseteq C_j$. Hence V is defined for $\langle P \sqcup Q \rangle$, and so $P \sqcup Q$ is raisable. Moreover, together with the fact that C_j is closed under union, it also follows from this result that $P+Q$ is raisable. Finally, we show that $\uparrow P$ is raisable if P is. Firstly, $\{V\langle P\rangle\} \in C_{n+1}$, and hence $\{V\langle P\rangle\}$ is raisable. If $P \cap M^D$ is empty, then $\uparrow P = \{V\langle P\rangle\}$, so $\uparrow P$ is raisable. If $P \cap M^D$ is non-empty, then $P \cap M^D \in C_n$, and so $P \cap M^D$ is raisable. Then by the above result, so is $\{V\langle P\rangle\} + (P \cap M^D) = \uparrow P$.

Constrained: Since $V(\gamma)$, when defined, is the multi-set underlying γ, and since γ and δ determine the same multi-set only if they determine the same set, it is immediate that $V(\gamma) = V(\delta)$ only if γ and δ determine the same set.

Intermediate: Since $V(\gamma)$ is the multi-set underlying γ, and $V(\delta)$ is the multi-set underlying δ, if the same multi-set underlies γ and δ, then $V(\gamma) = V(\delta)$. □

The canonical intermediate model of $\mathcal{L}_\approx$ is now defined as follows:

Definition 9. *The* canonical intermediate model $\mathcal{M}_I$ *of* $\mathcal{L}_\approx$ *is* $\langle M_I, V_I, [\cdot]_I\rangle$*, where for sentences* $A, B \in \mathcal{L}_\approx$

- $[A]_I = \langle\{A\}, \{\neg A\}\rangle$ *if is atomic*
- $[\neg A]_I = \neg[A]_I$
- $[A \wedge B]_I = [A]_I \wedge [B]_I$
- $[A \vee B]_I = [A]_I \vee [B]_I$

Equivalences that are true in the canonical intermediate model will also be called canonical.

Note that all contents assigned by $\mathcal{M}_I$ to some sentence are normal. For it is easy to see that all contents assigned to atomic sentences are normal, and so by the result that normality is preserved under truth-functional operations, it follows that all assigned contents are.

We now establish a correlation between the syntactic complexity of the formulas of the propositional language $\mathcal{L}_B$ and the level at which their contents are constructed in the recursive definition of the mode-space.

Definition 10. *For all* $P \in C$*, let rank*(P) *be the lowest* n *with* $P \in C_n$*.*

Lemma 12. *For all* $P \in \mathcal{R}$ *and* $Q, R \in \mathcal{R} \cap C^D$*:*

(i) $rank(Q \sqcup R) = max\{rank(Q), rank(R)\}$
(ii) $rank(Q + R) = max\{rank(Q), rank(R)\}$
(iii) $rank(\uparrow P) = rank(P) + 1$

Proof. For (i): Since each M_n is closed under fusion of modes, if $Q, R \in C_n$, then $Q \sqcup R \in C_n$, so the rank of $Q \sqcup R$ cannot be higher than the maximum rank of Q and R. Since each M_n is closed under non-empty submulti-sets, the rank of $Q \sqcup R$ also cannot be lower than the maximum rank of Q and R.

For (ii): In addition to the observations under (i), note that each C_n is closed under unions and non-empty subsets to see that the rank of $Q + R$ can be neither higher nor lower than the maximum rank of Q and R.

For (iii): Assume $rank(P) = n$. Firstly, by construction, $\{V\langle P\rangle\} = \{\lfloor P\rfloor\} \in C_{n+1}$.[57] Moreover, $\{\lfloor P\rfloor\} \notin C_n$. For suppose otherwise. Then $\lfloor P\rfloor \in M_n$. Now if $n = 0$, it follows that $\lfloor P\rfloor$ is a literal in $\mathcal{L}_{\approx}$, which it is not. If $n > 0$, it follows that $P \in C_{n-1}$, contrary to the supposition that $rank(P) = n$. So $rank(\{V\langle P\rangle\}) = n+1$. Now clearly, if $P \cap M^D$ is non-empty, $rank(P \cap M^D) \leq rank(P) = n$, and thus by part (ii), since $\uparrow P = \{V\langle P\rangle\} + (P \cap M^D)$, $rank(\uparrow P) = n+1$. If $P \cap M^D$ is empty, $\uparrow P = \{V\langle P\rangle\}$, so again $rank(\uparrow P) = n+1$. □

Definition 11. *We define in a simultaneous induction the* positive degree *$pdeg(A)$ and the* negative degree *$ndeg(A)$ of a formula $A \in \mathcal{L}_B$.*

- $pdeg(A) = ndeg(A) = 0$ *if A is atomic*
- $pdeg(\neg A) = ndeg(A)$
- $ndeg(\neg A) = pdeg(A) + 1$
- $pdeg(A \wedge B) = pdeg(A \vee B) = max\{pdeg(A), pdeg(B)\} + 1$
- $ndeg(A \wedge B) = ndeg(A \vee B) = max\{ndeg(A), ndeg(B)\} + 1$

The positive degree of a formula $A \in \mathcal{L}_B$ will sometimes also simply be called A's degree*, and denoted $deg(A)$. The degree of an* equivalence *$A \approx B$ is $max\{deg(A), deg(B)\}$.*[58]

Lemma 13. *In $\mathcal{M}_I$, for all $A \in \mathcal{L}_B$, $pdeg(A) = rank([A]^+_I)$ and $ndeg(A) = rank([A]^-_I)$.*

Proof. (For readability, I drop the subscript I.) By induction on the complexity of A. Suppose first that A is atomic. Then $pdeg(A) = ndeg(A) = 0$, and $[A]^+ = \{A\} \in C_0$, so $rank([A]^+) = 0$, and $[A]^- = \{\neg A\} \in C_0$, so $rank([A]^-) = 0$. Suppose now the thesis holds for A and B (IH). Then it also holds for $A \wedge B$, $A \vee B$, and $\neg A$. I give the proof for $pdeg(A \wedge B)$, the other cases are similar.

57 I write $\lfloor P, Q \ldots\rfloor$ for the multi-set including exactly $P, Q, \ldots$, each exactly as many times as it is listed.

58 We do this because the positive degree plays a somewhat more central role in the proofs to follow than does the negative degree. The asymmetry mirrors an asymmetry in the role or centrality of the positive content compared to the negative content of a formula. Since and insofar as ground is defined only in terms of positive content, negative content plays a somewhat lesser role. But negative content is still essential for the compositional definition of content, since the positive content of a negation is defined by appeal to the negative content of the negated formula. In an analogous way, the notion of the negative degree of a formula is essential since the positive degree of a negation is defined in terms of the negative degree of the negated formula. Thanks here to a referee for pressing me to clarify this.

$$\begin{aligned}
rank([A \wedge B]^+) &= rank(\uparrow[A]^+ \sqcup \uparrow[B]^+) && \text{by df. } \wedge \\
&= max\{rank(\uparrow[A]^+), rank(\uparrow[B]^+)\} && \text{by lemma 12(i)} \\
&= max\{rank([A]^+)+1, rank([B]^+)+1\} && \text{by lemma 12(iii)} \\
&= max\{rank([A]^+), rank([B]^+)\}+1 && \\
&= max\{pdeg(A), pdeg(B)\} && \text{by IH} \\
&= pdeg(A \wedge B) && \text{by df. } pdeg
\end{aligned}$$

□

For our purposes, the most important bit is the immediate corollary that only equivalences between formulas of equal degree are canonical:

Corollary 14. *For all sentences $A, B \in \mathcal{L}_{\approx}$, if $\mathcal{M}_I \models A \approx B$, then $deg(A) = deg(B)$.*

We are now in a position to prove completeness.

Theorem 15. (Completeness of the Intermediate System) $\vdash_{\mathfrak{I}} \varphi$ *whenever* $\models_{\mathfrak{I}} \varphi$.

Proof. As indicated earlier, we prove this by showing that every canonical equivalence is derivable. So assume that φ is canonical. Suppose first that $deg(\varphi) = 0$. Then φ has one of these forms, with A and B atomic:

(i) $A \approx B$
(ii) $A \approx \neg B$
(iii) $\neg A \approx B$
(iv) $\neg A \approx \neg B$

Cases (ii) and (iii) cannot obtain, for $[A]^+ = \{A\} \neq \{\neg B\} = [B]^- = [\neg B]^+$, and likewise $[\neg A]^+ = [A]^- = \{\neg A\} \neq \{B\} = [B]^+$. Equivalences of forms (i) and (iv) are canonical only if $A = B$, so they take the forms $A \approx A$ and $\neg A \approx \neg A$, respectively. But equivalences of these forms are derivable by (Reflexivity).

So suppose that equivalences of degree $\leq n$ are derivable if canonical, and suppose φ is of degree $n+1$. Then φ is an equivalence between two formulas of degree $n+1$. Each of them can be either a conjunction, or a disjunction, or a negated conjunction or disjunction, or a double negation. However, given (Symmetry) we need not separately consider, say, the case of $A \wedge B \approx C \vee D$ and that of $A \vee B \approx C \wedge D$. Moreover, given the DeMorgan equivalences, we also need not consider equivalences with a negated conjunction or a negated disjunction, since these cases may be reduced using the DeMorgan rules to cases of disjunctions or conjunctions of negations. So the cases we need to consider are these:

(i) $A \wedge B \approx C \wedge D$
(ii) $A \vee B \approx C \vee D$

(iii) $A \wedge B \approx C \vee D$
(iv) $A \wedge B \approx \neg\neg C$
(v) $A \vee B \approx \neg\neg C$

Case (i): By lemma 6(iv), if $A \wedge B \approx C \wedge D$ is canonical, then so are either (a) $A \approx C$ and $B \approx D$ or (b) $A \approx D$ and $B \approx C$. These equivalences are at most degree n, and so by IH, they are derivable. If (a), then by (Preservation $\wedge$), $A \wedge B \approx C \wedge B$ and $C \wedge B \approx C \wedge D$ are derivable. By (Transitivity), $A \wedge B \approx C \wedge D$ is derivable. If (b), then by (Preservation $\wedge$) we obtain $A \wedge B \approx D \wedge B$ and $B \wedge D \approx C \wedge D$. By (Commutativity $\wedge$) and (Transitivity), we may again derive $A \wedge B \approx C \wedge D$.

Case (ii): By lemma 6(vi), there are five ways for $A \vee B \approx C \vee D$ to be canonical. The first is that as in case (i), either (a) $A \approx C$ and $B \approx D$ or (b) $A \approx D$ and $B \approx C$ are canonical. By IH, these will be derivable, and similarly as before, using (Preservation $\vee$) and (Commutativity $\vee$) in place of the corresponding rules for conjunctions, we may derive $A \vee B \approx C \vee D$. The other four cases exhibit a common structure, so I shall confine myself to treating one of them, which is that $A \approx C$, $B \leq A$, and $D \leq C$ are canonical. Assuming that this is not also an instance of the first case, it follows that $B \approx D$ is not canonical. We can now show that both $[B]^+ \leq [D]^+$ and $[D]^+ \leq [B]^+$, which entails that $B \approx D$ is canonical, contrary to assumption. For since $[A]^+ = [C]^+$, for any $m \in [C]^+$, $m \sqcup \lfloor [B]^+ \rfloor \in [A \vee B]^+ = [C \vee D]^+$. By construction of the canonical model, no mode in $[C]^+$ contains any content more than finitely many times, so $m \sqcup \lfloor [B]^+ \rfloor$ is always distinct from m. Since $[C]^+$ moreover includes only finitely many modes, $m \sqcup \lfloor [B]^+ \rfloor \notin [C]^+$. It follows that $\lfloor [B]^+ \rfloor \in [D]^+$ and therefore $[B]^+ \leq [D]^+$. Similarly, for any $m \in [A]^+$, $m \sqcup \lfloor [D]^+ \rfloor \in [C \vee D]^+ = [A \vee B]^+$. By analogous reasoning as before, $\lfloor [D]^+ \rfloor \in [B]^+$ and hence $[D]^+ \leq [B]^+$.

The remaining cases (iii)–(v) cannot obtain. For cases (iii) and (iv), it suffices to note that both $[\neg\neg C]^+$ and $[C \vee D]^+$ always include the mode corresponding to the multi-set including only $[C]^+$, and exactly once, whereas every mode in a conjunction corresponds to a multi-set which either contains at least two elements, or contains one element at least twice.

For case (v), by lemma 6(iii), if $A \vee B \approx \neg\neg C$ is canonical, so is either $A \approx C$ or $B \approx C$. So suppose $[A]^+ = [C]^+$; the other case is analogous. Then whenever $m \in [C]^+$, $[A \vee B]^+$ also includes $m \sqcup \lfloor [B]^+ \rfloor$. As before, no mode in $[C]^+$ contains any content more than finitely many times, so $m \sqcup \lfloor [B]^+ \rfloor$ is always distinct from m, and since $[C]^+$ includes only finitely many modes, $[A \vee B]^+$ and $[C]^+$ are distinct. □

A.4. **Adequacy of the Extensional System.** For φ an equivalence in $\mathcal{L}_{\approx}$, we write $\vdash_{\mathfrak{E}} \varphi$ to say that φ is derivable within $\mathfrak{E}$, and we write $\models_{\mathfrak{E}} \varphi$ to say that φ is valid in the class of models based on complete, constrained, extensional mode-spaces. We show first that $\mathfrak{E}$ is sound with respect to that class of models.

Theorem 16. (Soundness of the Extensional System)
For every equivalence $\varphi \in \mathcal{L}_{\approx}$, if $\vdash_{\mathfrak{E}} \varphi$, then $\models_{\mathfrak{E}} \varphi$.

Proof. Given the previous soundness result in theorem 8 and the fact that every extensional mode-space is also intermediate, it suffices to establish soundness for the additional rules in $\mathfrak{E}$, i.e. (Collapse $\wedge/\vee$), (Collapse $\vee/\neg\neg$), (Introduction $\leq\wedge$), and (Introduction $\leq\vee$). The soundness of (Collapse $\wedge/\vee$), (Collapse $\vee/\neg\neg$) is immediate from lemma 5(ii)-(iii). The soundness of (Introduction $\leq\wedge$) and (Introduction $\leq\vee$) is straightforward given lemma 7 and the principles ($<\wedge$) and ($<\vee$) in lemma 1. □

The completeness proof proceeds in close analogy to that for the semi-extensional system. We first define the canonical extensional mode-space, simply replacing any reference to multi-sets in the definition of the canonical semi-extensional mode-space by reference to the corresponding set.

Definition 12. *The* canonical extensional mode-space *for $\mathcal{L}_{\approx}$ is the pair $\langle M_E, V_E\rangle$, where*

- $M_0 := \{A \in \mathcal{L}_{\approx} : A$ *is a literal*$\}$
- $C_n := \wp(M_n)\backslash\{\emptyset\}$
- $M_{n+1} := \{m : m \in M_0$ *or m is a non-empty set of members of* $C_n\}$
- $C := \bigcup_{n\in N} C_n$
- $M_E := \{m : m \in M_0$ *or m is a non-empty set of members of* $C\}$
- $V_E(\gamma) = \Gamma$ *if γ is a non-empty countable sequence of members of C and Γ is the underlying set.*
- $V_E(\gamma)$ *is undefined otherwise.*

By straightforward adjustments to the earlier proof, it may be shown that $\langle M_E, V_E\rangle$ is a mode-space of the desired kind.

Lemma 17. *$\langle M_E, V_E\rangle$ is complete, constrained, and extensional mode-space.*

The canonical extensional model of $\mathcal{L}_{\approx}$ is defined in the obvious way:

Definition 13. *The* canonical extensional model $\mathcal{M}_E$ *of $\mathcal{L}_{\approx}$ is $\langle M_E, V_E, [\cdot]_E\rangle$, where for sentences $A, B \in \mathcal{L}_{\approx}$*

- $[A]_E = \langle\{A\}, \{\neg A\}\rangle$ *if is atomic*
- $[\neg A]_E = \neg[A]_E$
- $[A \wedge B]_E = [A]_E \wedge [B]_E$

- $[A \vee B]_E = [A]_E \vee [B]_E$

The lemmata concerning the correspondence of the degree of syntactic complexity of an $\mathcal{L}_{\approx}$-sentence of the rank in the hierarchy of propositions in the construction of the mode-space unproblematically carries over to the extensional setting, so we again obtain the desired corollary:

Corollary 18. *For all sentences $A, B \in \mathcal{L}_{\approx}$, if $\mathcal{M}_E \models A \approx B$, then $deg(A) = deg(B)$.*

In preparation of the completeness proof, it helps to first prove the following lemma.

Lemma 19. *If sentences A, $B \in \mathcal{L}_B$ are degree $\leq n$, and if every equivalence up to and including degree n is derivable if canonical, then $B \leq A$ is also derivable if canonical.*

Proof. Assume the antecedent. By definition of $\leq$, $B \leq A$ is canonical just in case either $B \approx A$ is canonical or $[B]^+ < [A]^+$. By assumption, if $B \approx A$ is canonical, then it is derivable. But then by (Preservation $\vee$), so is $A \vee B \approx A \vee A$, which is $B \leq A$. So suppose that $[B]^+ < [A]^+$. Then $deg(A) > 0$, and so A takes one of these forms

(a) $D \vee E$
(b) $D \wedge E$
(c) $\neg\neg D$
(d) $\neg(D \vee E)$
(e) $\neg(D \wedge E)$

where D and E are degree $< n$.

If (a), and thus $[B]^+ < [D \vee E]^+ = [D]^+ \vee [E]^+$, it follows that either $[B]^+ \leq [D]^+$ or $[B]^+ \leq [E]^+$ and hence that either (i) $B \leq D$ is canonical or (ii) $B \leq E$ is canonical. Since D and E are degree $< n$, $D \vee D$ and $E \vee E$ are degree $\leq n$, so by corollary 18, the equivalences $B \leq D$ and $B \leq E$ are degree $\leq n$. So by assumption, if (i), then $B \leq D$ is derivable, and if (ii), then $B \leq E$ is derivable. Suppose (i). Then by (Introduction $\leq\vee$), $B \leq D \vee E$ is derivable, which is $B \leq A$. Suppose (ii). Then by (Introduction $\leq\vee$), $B \leq E \vee D$ is derivable. Using (Commutativity $\vee$), (Transitivity), and (Preservation $\vee$), we may derive from this $B \leq D \vee E$, that is $B \leq A$.

If (b), and thus $\{[B]^+\} \in [D \wedge E]^+ = [D]^+ \wedge [E]^+$, it follows that $[B]^+ \leq [D]^+$ and $[B]^+ \leq [E]^+$ and hence that both $B \leq D$ and $B \leq E$ are canonical. As before, these equivalences are degree $\leq n$ and thus derivable. By (Introduction $\leq\wedge$), so is $B \leq D \wedge E = A$.

The remaining cases can be reduced to the previous ones using the DeMorgan identities and lemma 5. For illustration, suppose that case (c) obtains. Then $[B]^+ < [\neg\neg D]^+$. But $[\neg\neg D]^+ = [D \vee D]^+ = [D]^+ \vee [D]^+$. By the reasoning in case (a), $B \leq D \vee D$ is derivable. Using the derivable equivalence of $D \vee D$ to $\neg\neg D$, we may derive $B \leq \neg\neg D$, i.e. $B \leq A$. □

Theorem 20. (Completeness of the Extensional System)
For every equivalence $\varphi \in \mathcal{L}_{\approx}$, *if* $\models_{\mathfrak{E}} \varphi$, *then* $\vdash_{\mathfrak{E}} \varphi$.

Proof. We show by induction on the degree of equivalences that every canonical equivalence is derivable. Suppose φ is a canonical equivalence. The case of $deg(\varphi) = 0$ is exactly as in the semi-extensional case.

Now assume all canonical equivalences of degree $\leq n$ are derivable and suppose φ is of degree $n+1$. So φ is an equivalence between two formulas of degree $n+1$. Each of them can be either a conjunction, or a disjunction, or a negated conjunction or disjunction, or a double negation. The last three cases can be reduced to the first two in the same way we did in the proof of lemma 19. So we only have three kinds of equivalences of degree $n+1$ to consider, namely instances of the following forms, where A, B, C, and D are each of some degree $\leq n$:

(i) $A \wedge B \approx C \wedge D$
(ii) $A \vee B \approx C \vee D$
(iii) $A \wedge B \approx C \vee D$

Case (i): By lemma 6(iv), if $A \wedge B \approx C \wedge D$ is canonical, then so are either (a) both $A \approx C$ and $B \approx D$, or (b) both $A \approx D$ and $B \approx C$. So suppose (a). The equivalences $A \approx C$ and $B \approx D$ are both at most degree n, so by IH, $A \approx C$ and $B \approx D$ are derivable. Using (Preservation $\wedge$), $A \wedge B \approx C \wedge B$ and $C \wedge B \approx C \wedge D$ are derivable. Using (Transitivity), $A \wedge B \approx C \wedge D$ is derivable. Now suppose (b) $A \approx D$ and $B \approx C$ are canonical. Then these are at most degree n and thus derivable. Using (Preservation $\wedge$), $A \wedge B \approx D \wedge B$ and $B \wedge D \approx C \wedge D$ are derivable. Using (Commutativity $\wedge$) and (Transitivity), $A \wedge B \approx C \wedge D$ is derivable.

Case (ii): By lemma 6(vi), if $A \vee B \approx C \vee D$ is canonical, there are five ways this can come about. One is that $A \wedge B \approx C \wedge D$ is canonical, in which case as before, either $A \approx C$ and $B \approx D$ are canonical, or $A \approx D$ and $B \approx C$ are canonical. These will then be derivable, and much as in case (i) but using (Commutativity $\vee$) instead of (Commutativity $\wedge$), $A \vee B \approx C \vee D$ is derivable from them. A second way in which $A \vee B \approx C \vee D$ can be canonical is by $A \approx C$, $B \leq A$, and $D \leq A$ being canonical; the remaining cases are analogous and will be omitted. Then by IH and lemma 19, $A \approx C$, $B \leq A$ and $D \leq A$ are all derivable. From these, using mainly (Commutativity $\vee$) and (Preservation $\vee$), we may then derive $A \vee B \approx C \vee D$.

Case (iii): By lemma 6(v), if $A \wedge B \approx C \vee D$ is canonical, then so is $A \approx B$, which, by IH, is derivable. But then also $[A \wedge B]^+ = [A]^+ \wedge [B]^+ = [A]^+ \wedge [A]^+ = [A]^+ \vee [A]^+ = [A \vee A]^+$, so $A \vee A \approx C \vee D$ is also canonical, and by case (ii) derivable. From these, using mainly (Collapse $\wedge/\vee$) and (Preservation $\wedge$), we may derive $A \wedge B \approx C \vee D$. □

Appendix B. Comparison of Deductive Systems

Theorem 21. *For every equivalence $\varphi \in \mathcal{L}_{\approx}$, if $\vdash_{\mathfrak{R}} \varphi$ then $\vdash_{\mathfrak{I}} \varphi$.*

Proof. Call a theorem φ of $\mathfrak{R}$ *unproblematic* if the theorem produced by applying the rule (Pres.$\neg$) to φ can also be derived within $\mathfrak{I}$. We show by an induction on the length of derivations that all theorems of $\mathfrak{R}$ are unproblematic. From this it follows straightforwardly that all theorems of $\mathfrak{R}$ are theorems of $\mathfrak{I}$. Consider first the case of a derivation D of length 1. There are three case:

1. D consists in an application of (Reflexivity). Then the result of applying (Preservation $\neg$) can also be achieved simply by an application of (Reflexivity).
2. D consists in an application of (Commutativity $\vee$), so the theorem established by D is $A \vee B \approx B \vee A$. Application of (Preservation $\neg$) yields $\neg(A \vee B) \approx \neg(B \vee A)$. This can be derived within $\mathfrak{I}$ from $A \vee B \approx B \vee A$ by application of the DeMorgan rules and (Commutativity $\wedge$): $(\neg(A \vee B) \approx \neg A \wedge \neg B \approx \neg B \wedge \neg A \approx \neg(B \vee A))$
3. D consists in an application of (Commutativity $\wedge$). Analogous to the previous case.

Suppose then that derivations up to length n produce only unproblematic theorems, and suppose D has length $n+1$. The cases in which the final step in D consists in the application of one of the premise-less rules just discussed are exactly as before. The remaining cases are five, according as the final step in D is an application of

1. (Symmetry) Then application of (Preservation $\neg$) produces $\neg B \approx \neg A$. By IH, $\neg A \approx \neg B$ can be derived within $\mathfrak{I}$, and thus by (Symmetry), so can $\neg B \approx \neg A$.
2. (Transitivity) Then application of (Preservation $\neg$) produces $\neg A \approx \neg C$. By IH, $\neg A \approx \neg B$ and $\neg B \approx \neg C$ can be derived within $\mathfrak{I}$, and thus by (Transitivity), so can $\neg A \approx \neg C$.
3. (Preservation $\vee$) Then application of (Preservation $\neg$) produces $\neg(A \vee C) \approx \neg(B \vee C)$. By IH, $\neg A \approx \neg B$ can be derived within $\mathfrak{I}$. By (Preservation $\wedge$), $\neg A \wedge \neg C \approx \neg B \wedge \neg C$ can then be derived, and by DeMorgan, so can $\neg(A \vee C) \approx \neg(B \vee C)$.
4. (Preservation $\wedge$) Analogous to the previous case.
5. (Preservation $\neg$) Then application of (Preservation $\neg$) produces $\neg\neg A \approx \neg\neg B$, which can be derived within $\mathfrak{I}$ from $A \approx B$ by (Preservation $\neg\neg$).

□

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